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TECHNIQUES OF APPLIED MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS AND GREEN'S FUNCTIONS

by

BERNARD FRIEDMAN

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Ordinary Differential Equations and Green's Functions

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Abstract

This report continues the investigation of Research Report No. EM-47 on the applications of recently developed mathematical techniques. The aim here is to present the theory of Green's functions for ordinary differential equations and for systems of ordinary differential equations. This theory can be used to obtain the spectral representation of ordinary differential equations and to solve partial differential equations.

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Introduction

In this report the methods previously developed† will be used to discuss the solutions of ordinary linear differential equations. The main problem will be to solve

$$Lu = f, \tag{3.1}$$

where L is a differential operator, f a known function, and u the unknown function. The general theory of linear equations tells us that there are two important approaches to solving (3.1). One approach is to find the inverse operator to L, that is, to find an operator L^{-1} such that $L^{-1}L$ is the identity operator. The other approach is to find the spectral representation of L by studying the solutions of the equations

$$Lu = \lambda u,$$

where λ is an arbitrary constant. Of course, L is not a finite-dimensional bounded operator, so the theorems of matrix theory are not immediately valid. We shall use them, however, to suggest the theorems which we shall try to prove.

Both approaches, that of finding the inverse operator and that of finding its spectral representation, will be used in discussing differential operators. Most of the discussion will be devoted to the consideration of special examples in order to reveal the techniques and the possible results. After the treatment of each group of special examples, there will be a section giving a general discussion of the methods employed in that particular type of problem.

The Linear Vector Space and its Operators

The linear vector space that will be used is the function space of all real functions which are Lebesgue square integrable. At first, we shall restrict ourselves to functions defined over a finite interval, which for simplicity we shall always assume to be the interval (0,1). Our space $\mathcal{S}$ is then the space of all functions u(x) such that

$$\int_0^1 u(x)^2 dx < \infty.$$

If u and v belong to $\mathcal{S}$, then we define

† B. Friedman, 'Techniques of Applied Mathematics - Theory of Distributions,' New York University, Mathematics Research Group, Research Report No. EM-47, October, 1952.

$$(3.2) \qquad (u,v) = \int_0^1 u(x)v(x)dx.$$

For some applications, particularly to the theory of Bessel functions, it is convenient to modify the definition of the space $\mathcal{S}$ and the scalar product by introducing a non-negative "weight" function $w(x)$. $\mathcal{S}$ is now the space of all functions $u(x)$ such that

$$\int_0^1 u(x)^2 w(x)dx < \infty,$$

and the scalar product becomes

$$(u,v) = \int_0^1 u(x)v(x)w(x)dx.$$

We shall be interested in linear differential operators such as $\frac{d}{dx}$ or $x\frac{d^2}{dx^2} + \frac{d}{dx}$ acting on $\mathcal{S}$. These operators cannot be applied to every function in $\mathcal{S}$ because, first, the function may not be differentiable and, second, even if the function is differentiable, the derivative may not belong to $\mathcal{S}$.

Let L be the following linear differential operator:

$$p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots p_{n-1}(x)\frac{d}{dx} + p_n(x).$$

Here $p_0(x)$, $p_1(x),\cdots,p_n(x)$ are assumed to be piecewise continuous functions. Occasionally, we shall permit $p_n(x)$ to be a delta function. In a later section we shall discuss the precise meaning of the differential operator in the case where the coefficients are discontinuous functions or delta functions. The set of functions $u(x)$ in L such that Lu exists and belongs to $\mathcal{S}$ will be called the domain of L. It is easy to show that the domain is a linear manifold in $\mathcal{S}$. However, for many applications, the domain is too large a manifold, since it always contains an infinite number of solutions of the equation

$$Lu = f(x).$$

If this equation is to have a unique solution, initial or boundary conditions must be imposed on u; there will be n such conditions if L is an n-th order differential operator.

Put

$$(3.3) \qquad B_k(u) = \sum_{j=0}^{n-1} a_{kj}u^{(j)}(0) + \sum_{j=0}^{n-1} b_{kj}u^{(j)}(1)$$

where $u^{(j)}(x)$ represents the j-th derivative of u if $j \geq 1$ and where

$u^{(0)}(x) = u(x)$ if $j = 0$. The equations

$$B_k(u) = 0, \quad k = 1, 2, \cdots, n \tag{3.3}$$

will be n linear homogeneous initial or boundary conditions. The conditions are linear since u and its derivatives appear only in the first degree; they are homogeneous because if u and its derivatives are all multiplied by a constant c, the form $B_k(u)$ is multiplied by the same constant c. The conditions will be initial conditions if all $b_{kj} = 0$; they will be end conditions if all $a_{kj} = 0$; and otherwise, we will call them boundary conditions. For our present purposes this distinction is immaterial. We must assume, however, that the conditions (3.3) are independent, that is, there do not exist constants c_1, c_2, $\cdots$, c_n such that

$$c_1B_1(u) + c_2B_2(u) + \cdots c_nB_n(u) = 0$$

for all functions u(x).

The conditions (3.3) determine a linear manifold, $\mathcal{M}$, in $\mathcal{S}$. It is to be noted that $\mathcal{M}$ is not a closed subspace, that is, there may exist in $\mathcal{M}$ a sequence of functions $u_n(x)$ which converge to a limit u(x) in $\mathcal{S}$, but u(x) is not in $\mathcal{M}$. For example, let $\mathcal{M}$ be defined by the condition $u(0) = 0$. Then consider the sequence of functions

$$\begin{aligned} u_n(x) &= 0, \quad x < 1/n \\ &= 1, \quad x > 1/n. \end{aligned}$$

It is clear that the sequence $u_n(x)$ converges in the mean to the limit function $u(x) = 1$, but the limit function does not itself belong to $\mathcal{M}$.

Logically, we should consider L as the differential operator restricted to that part of its domain which also belongs to $\mathcal{M}$, so that the same differential operator with different sets of initial or boundary conditions represents different linear operators. However, for convenience in writing, we shall use L to represent the formal differential operator only and use L and the appropriate manifold to represent the complete linear operator.

It will occasionally be useful to consider a system of n first-order differential equations instead of one n-th order differential equation. In such a case the appropriate space is the space of vectors U with components $u_1(x)$, $u_2(x)$,$\cdots$,$u_n(x)$, where each component is a real square integrable function. The appropriate scalar product is

$$(V,U) = \int_0^1 \left[u_1 v_1 + u_2 v_2 + \cdots + u_n v_n\right] dx.$$

Note that this is a combination of the scalar product in an n-dimensional vector space with the scalar product in a space of functions of one variable.

The differential operators in this space can be any expression of the form

$$\frac{d^k}{dx^k} + A_1 \frac{d^{k-1}}{dx^{k-1}} + \cdots + A_k$$

where now $A_1, \cdots, A_k$ are matrices whose elements are piecewise continuous functions of x. We shall usually restrict ourselves to the consideration of the simplest such operator, namely

$$\frac{d}{dx} - A;$$

consequently, a typical differential equation would be

$$\frac{dU}{dx} = AU. \tag{3.4}$$

Here A is a matrix with elements $a_{ij}(x)$ $(i,j = 1,2,\cdots,n)$.

When equation (3.4) is written in full, it looks like this:

$$\frac{du_1}{dx} = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{du_n}{dx} = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n.$$

The fact that we consider only first-order differential operators is not as restrictive as it may appear, because we shall now show that every n-th order differential equation

$$u^{(n)} + p_1(x)u^{(n-1)} + \cdots + p_n(x)u = 0$$

can be written in the form (3.4). Put

$$u_1(x) = u(x),\ u_2(x) = u'(x), \cdots,\ u_n(x) = u^{(n-1)}(x)$$

and put

$$(3.5) \qquad A = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 0 & 1 \\ -p_n & -p_{n-1} & \cdot & \cdot & \cdot & -p_2 & -p_1 \end{pmatrix};$$

then (3.4) is equivalent to the given n-th order differential equation.

Similarly, it can be shown that a system of n-th order differential equations may be reduced to a system of first-order differential equations.

Of course, the differential operator in (3.4) is not completely defined until we specify the boundary conditions on U. Since (3.4) is a first-order system, we should expect one vector condition on the values of U at $x = 0$ and $x = 1$. Since one vector condition implies n scalar conditions (one for each component), we may write the conditions on U as follows:

$$B_1(U) = B_2(U) = \cdots B_n(U) = 0.$$

Here, $B_i(U)$, $(i = 1,2,\ldots,n)$ is a linear scalar expression containing the values of $u_1, u_2, \cdots, u_n$ at $x = 0$ and $x = 1$.

Adjoint of a Differential Operator: Hermitian Operators

It is important to consider simultaneously a linear operator **L** and its adjoint L^*. The adjoint is defined by considering the equation

$$(3.6) \qquad (v,Lu) = (w,u)$$

and putting

$$w = L^*v.$$

For differential operators, an equation similar to (3.6) can be obtained by integration by parts. For example, let L be $\frac{d}{dx}$ on the manifold $\mathcal{M}$ defined by the condition $u(0) = 2u(1)$. Then

$$(v,Lu) = \int_0^1 v\,\frac{du}{dx}\,dx = vu\,\Big|_0^1 - \int_0^1 u\,\frac{dv}{dx}\,dx$$

$$= u(1)\left[v(1)-2v(0)\right] - \int_0^1 u\,\frac{dv}{dx}\,dx = (w,u). \tag{3.7}$$

We see that L^* consists of two parts: a differential operator, $-\frac{d}{dx}$, and some boundary terms. The differential operator $-\frac{d}{dx}$ is called the formal adjoint to the differential operator $\frac{d}{dx}$. The adjoint to L on the manifold $\mathcal{M}$ will be the formal adjoint $-\frac{d}{dx}$ on the manifold defined by the condition $v(1) = 2v(0)$. Now

$$(v,Lu) = (L^*v,u)$$

where

$$L^*v = -\frac{dv}{dx}$$

and where v satisfies the condition $v(1) = 2v(0)$.

It is to be noted that in this example L acts on the manifold of square integrable functions $u(x)$ such that $u(0) = 2u(1)$, while L^* acts on the manifold of square integrable functions $v(x)$ such that $v(0) = v(1)/2$. This is an illustration of the general situation where the manifold on which L^* acts may be different from the manifold on which L acts. We call one such manifold the dual of the other.

If $L = L^*$, the differential operator is said to be formally self-adjoint. If, in addition, the boundary conditions for L and L^* are equivalent, that is, if they define the same linear manifold, then the operator is said to be self-adjoint. As an illustration, suppose

$$L = e^x\frac{d^2}{dx^2} + e^x\frac{d}{dx}$$

on the manifold defined by the conditions

$$u'(0) = 0, \quad u(1) = 0.$$

Then

$$(v,Lu) = \int_0^1 v(e^xu'' + e^xu') = \int_0^1 v(e^xu')'dx$$

$$= \left[ve^xu'\right]_0^1 - \int_0^1 (v'e^x)u'dx$$

$$= \left[ve^xu'\right]_0^1 - \left[v'e^xu\right]_0^1 + \int_0^1 u(v'e^x)'dx \tag{3.8}$$

$$= u'(1)v(1)e^1 + v'(0)u(0) + \int_0^1 u(e^xv'' + e^xv')dx\,.$$

This shows that

$$L^* = e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx} .$$

In order that

$$(v,Lu) = (L^*v,u)$$

for all u, the boundary conditions that v satisfies must be

$$v'(0) = 0, \quad v(1) = 0.$$

Since $L = L^*$, the differential operator is formally self-adjoint, and since the boundary conditions on u(x) and v(x) are the same, the differential operator is self-adjoint.

The adjoint for the general differential operator of n-th order

$$L = p_o(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$$

which acts on the manifold defined by the conditions

$$B_k(u) = 0, \quad k = 1,2,\cdots,n,$$

can be found in the same way. Integration by parts shows that

$$\int_a^b vLu\,dx = \int_a^b v\left[p_o u^{(n)} + \cdots p_n u\right] dx$$

$$= \int_a^b u\left[p_n v - (p_{n-1}v)' + \cdots (-)^n (p_o v)^{(n)}\right] dx + J(v,u)\Big|_a^b . \tag{3.9}$$

Here J(v,u) represents the terms that are produced during the integration by parts. For example, in (3.7)

$$J(v,u)\Big|_o^1 = u(1)v(1) - u(0)v(0)$$

while in (3.8)

$$J(v,u)\Big|_o^1 = \left[ve^x u' + ve^x u\right]_o^1 - u(ve^x)'\Big|_o^1 .$$

We shall call J(u,v) the conjunct† of the functions u and v. It is clear that J(u,v) is a linear homogeneous function of u and v and their derivatives.

---

† Ince, H.L., in his book, Ordinary Differential Equations (Dover, 1944), uses the term "bilinear concomitant".

Equation (3.9) may be written as follows:

$$\int_0^1 vLu\,dx = \int_0^1 uL^*v\,dx + J(v,u)\Big|_0^1 ,$$

where L^*v is the differential operator defined by the following formula:

$$(3.10) \qquad L^*v = p_n v - (p_{n-1}v)' + \cdots (-)^n (p_o v)^{(n)}.$$

Now, all that is needed to complete the definition of the adjoint of L is to determine the manifold on which L^* acts. This is done by requiring $J(u,v)\Big|_0^1$ to vanish when u is any function in the manifold on which L acts and v is any function in the manifold on which L^* acts. We know that the manifold on which L acts is determined by the boundary conditions on u, $B_k(u) = 0$. These conditions are n linear equations connecting the 2n numbers which are the values of u(x) and its first (n-1) derivatives at $x = 0$ and $x = 1$. From these linear equations, n of the numbers, say $u(0)$, $u'(0)$, $\cdots u^{(k)}(0)$, $u(1)$, $u'(1)$, $\cdots u^{(n-k-2)}(1)$, can be expressed in terms of the n numbers remaining, namely $u^{(k+1)}(0)$, $\cdots$, $u^{(n-1)}(0)$, $u^{(n-k-1)}(1)$, $\cdots u^{(n-1)}(1)$. Substitute these expressions in $J(v,u)\Big|_0^1$ and we have

$$u^{(k+1)}(0)B_1^*(v) + u^{(k+2)}(0)B_2^*(v) + \cdots + u^{(n-1)}(1)B_n^*(v),$$

where $B_1^*(v)$, $B_2^*(v)$, $\cdots$, $B_n^*(v)$ are n linear expressions involving v(x) and its first (n-1) derivatives at $x = 0$ and $x = 1$. Consequently the manifold on which L^* acts will be defined by the boundary conditions:

$$B_1^*(v) = B_2^*(v) = \cdots = B_n^*(v) = 0.$$

The examples worked out at the beginning of this section are illustrations of this method.

In the case of a system of first-order differential equations, the adjoint is obtained in a manner similar to that above. Suppose that U is a column vector whose n components are functions, and suppose that A is an n by n matrix whose elements are functions. Consider the following system of first-order differential equations :

$$(3.11) \qquad \frac{dU}{dx} = AU$$

with the boundary conditions

$$B_1(U) = B_2(U) = \cdots B_n(U) = 0.$$

Here, $B_k(U)$ $(k = 1,2,\cdots,n)$ is a linear scalar expression containing the values of the components of U at $x = 0$ and $x = 1$.

The system adjoint to (3.11) is obtained by taking the adjoint of the operators that appear in (3.11); consequently, if we suppose V is a column vector whose n components are functions, then the adjoint system is

$$-\frac{dV}{dx} = A^*V \tag{3.12}$$

where A^* is the matrix adjoint to A.

The adjoint system can also be obtained by an integration by parts, but to do this we must use the appropriate scalar product. Suppose that V' is the row vector whose components are those of V, and suppose that V'•U represents the ordinary scalar product of vectors, that is, the sum of the products of corresponding components; then, as we have defined before,

$$(V,U) = \int_0^1 V' \cdot U dx .$$

Now, put

$$LU = \frac{dU}{dx} - AU$$

and we have

$$(V,LU) = \int_0^1 V' \cdot \frac{dU}{dx}\, dx - \int_0^1 V' \cdot AU dx = V' \cdot U\Big|_0^1 - \int_0^1 \frac{dV'}{dx} \cdot U dx - \int_0^1 V'A \cdot U dx.$$

This shows that the adjoint system is

$$-\frac{dV'}{dx} - V'A = 0.$$

When this is re-written in terms of column vectors instead of row vectors it becomes

$$-\frac{dV}{dx} - A^*V = 0,$$

which we have already obtained previously in Equation (3.12).

The boundary conditions on V are found from the condition that the term which appeared in the integration by parts, namely $V' \cdot U\Big|_0^1$, should vanish identically for all vectors U that satisfy the boundary conditions. For example, suppose that U has the components $u_1(x)$, $u_2(x)$, and A is such that equation (3.11) becomes

$$\frac{dU}{dx} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} U;$$

suppose also that the boundary conditions are as follows:

(3.13) $$u_1(0) = u_2(1), \quad u_2(0) = 0.$$

Now, let V have the components $v_1(x)$, $v_2(x)$; then the equation satisfied by V is

$$\frac{dV}{dx} = - \begin{pmatrix} a & c \\ b & d \end{pmatrix} V.$$

To obtain the boundary conditions satisfied by V, we notice that

$$V'.U\Big|_0^1 = v_1u_1+v_2u_2\Big|_0^1 = v_1(1)u_1(1)+v_2(1)u_2(1)-v_1(0)u_1(0)-v_2(0)u_2(0)$$

(3.14) $$= v_1(1)u_1(1) + \left[v_2(1)-v_1(0)\right]u_1(0),$$

where the last result is obtained by the use of (3.13). It is clear that (3.14) will vanish identically for all vectors U if and only if

$$v_1(1) = 0, \quad v_2(1) = v_1(0).$$

These are the required boundary conditions on V.

The discussion of this section may be summarized in the following Method. To find the adjoint of a differential operator L in a space $\mathcal{S}$ consider the scalar product (v,Lu). With the help of integration by parts, consider it as the scalar product of u with some vector w, which depends on v. The transformation from v to w defines the adjoint operator L^*. The boundary conditions on v are determined by the requirement that the terms resulting from the integration by parts vanish.

It is to be noted that the form of the boundary conditions on v is not unique; for if $\beta_1^*(v) = \beta_2^*(v) = 0$ are any two boundary conditions on v then the conditions $\alpha_1\beta_1^*(v)+\alpha_2\beta_2^*(v) = \beta_1\beta_1^*(v)+\beta_2\beta_2^*(v) = 0$ are completely equivalent to the original ones, as long as $\alpha_1\beta_2-\alpha_2\beta_1 \neq 0$. Nevertheless, despite the fact that the boundary conditions are not unique, it is easy to show that the manifold defined by any set of boundary conditions is always the same.

Problems:

3.1 Find the adjoint differential operator L^* and the manifold on which it acts if

(a) $Lu = u'+a(x)u'+b(x)u$, where $u(0) = u'(1)$ and $u(1) = u'(0)$,

(b) $Lu = -(p(x)u')'+q(x)u$, where $u(0) = u(1)$ and $u'(0) = u'(1)$.

Assume that the scalar product is

$$(u,v) = \int_0^1 uv\, dx.$$

3.2 Suppose that $Lu = u''$, where $a_1u(0)+b_1u'(0)+c_1u(1)+d_1u'(1) = 0$ and $a_2u(0)+b_2u'(0)+c_2u(1)+d_2u'(1) = 0$. Find L^* and the manifold on which it acts. For what values of the constants $a_1, b_1, \cdots, c_2, d_2$ is the operator L self-adjoint?

3.3 Show that the differential operator

$$Lu = -\frac{1}{w(x)}(p(x)u')' + q(x)u$$

is formally self-adjoint if the scalar product is

$$(u,v) = \int_0^1 u(x)v(x)w(x)dx.$$

3.4 Show that the differential operator

$$Lu = a(x)u'' + b(x)u' + c(x)u$$

can be identified with the operator considered in Problem 3.3 by putting $q = c$, $p = \exp\int b/a\, dx$, $w = -a \exp\left[-\int b/a\, dx\right]$. It follows that the general second-order differential operator is formally self-adjoint if the appropriate scalar product is used.

3.5 Find the system adjoint to the following:

$$\frac{du}{dx} = \begin{pmatrix} p_1 & p_2 & 0 \\ 0 & q_1 & q_2 \\ r_1 & r_2 & 0 \end{pmatrix} u,$$

where $u_1(0)+u_2(0)+u_3(0) = 0$, $u_1(1) = u_2(1)$ and $u_3(1) = u_1(1)+u_2(1)$.

3.6 Consider the complex-type scalar product

$$(v,u) = \int_0^1 \overline{v(x)}\, u(x)w(x)dx,$$

where $w(x)$ is a non-negative real function. Is $Lu = -\frac{1}{w}(pu')'+qu$ formally self-adjoint in a space with the above scalar product? Distinguish between the case where p and q are both real functions and the case where p and q are complex functions.

Self-Adjoint Second-Order Differential Operators

Since a large number of the differential equations that occur in mathematical physics are of the second order, we shall give special consideration to second-order differential operators. From Problems 3.3 and 3.4 we see that any such operator is self-adjoint if the appropriate scalar product is used; therefore we shall write the operator as follows:

$$Lu = -\frac{1}{w}(pu')' + qu, \tag{3.15}$$

and we shall assume that the scalar product has $w(x)$ as its weight function.

The reason for the minus sign in the definition of L is that in certain cases it permits us to consider L as a positive-definite operator. A linear operator L is called positive-definite if $(u,Lu) > 0$ for all values of u except $u = 0$. Now, when L is defined by (3.15), we have

$$(u,Lu) = \int_0^1 u\left[-\frac{1}{w}(pu')' + qu\right] w\,dx$$

$$= \int_0^1 (pu'^2 + qwu^2)\,dx - puu'\Big|_0^1 ;$$

consequently, if $p > 0$, $q > 0$, $w > 0$, and if the boundary conditions on u are such as to make the last term on the right vanish, then L will be a positive definite operator.

So far, L is only formally self-adjoint. Under what conditions will L be actually self-adjoint? To decide this, consider the difference $(v,Lu) - (Lv,u)$. We have

$$(v,Lu) - (Lv,u) = J(v,u)\Big|_0^1 = -p(x)(vu'-uv')\Big|_0^1 . \tag{3.16}$$

L will be self-adjoint if the conjunct J vanishes identically when u and v are in the same manifold. We shall not discuss the most general conditions for which this happens (see Problem 3.2), but instead shall consider two cases which are important for applications:

1) The boundary conditions are unmixed if each boundary condition involves the values of u and its derivatives either at $x = 0$ or at $x = 1$ but not at both. The typical unmixed boundary condition is

$$\alpha_0 u(0) + \beta_0 u'(0) = 0.$$

It is easy to show that if u satisfies an unmixed boundary condition at x = 0 and an unmixed boundary condition at x = 1, then L is self-adjoint.

2) The boundary conditions are <u>periodic</u> if they have the form:

$$u(0) = u(1),\ u'(0) = u'(1).$$

In this case again it is easy to show that L is self-adjoint.

We now prove

Theorem I. <u>If u is any solution of the equation Lu = 0 and if v is any solution of the equation $L^*v = 0$, then the conjunct of u and v is a constant whose value depends on u and v.</u>

Here L and L^* are formally adjoint differential operators in the full space $\mathcal{S}$. The proof is an immediate consequence of the definition of the conjunct. From equation (3.9) we have

$$\int_a^b vLu\ dx - \int_a^b uL^*v\ dx = J(u,v)\ \Big|_a^b .$$

From the hypothesis on u and v it follows that

$$J(v,u)\ \Big|_a^b = 0;$$

consequently the value of $J(v,u)$ at x = b is the same as its value for x = a. Since a and b are arbitrary points, this proves that the value of $J(v,u)$ is constant.

Corollary. <u>If L is a formally self-adjoint operator and if u_1 and u_2 are any solutions of Lu = 0, then the conjunct of u_1 and u_2 is a constant whose value depends on the functions u_1 and u_2.</u>

Note that the theorem and corollary are also valid for an n^{th} order system.

As an illustration of this theorem consider the following differential equation:

(3.17) $$u'' + u = 0.$$

Since (3.17) is formally self-adjoint, v will also be a solution of it. The conjunct of v and u is now

$$J(v,u) = vu' - uv' .$$

Theorem I states that this expression is constant for any two solutions of (3.16). For example, if u = sinx and v = cosx, then

$$J(v,u) = \cos x \cos x - \sin x(-\sin x) = \text{constant}.$$

This is obviously correct because

$$\cos^2 x + \sin^2 x = 1$$

for all values of x.

The Wronskian of the Solutions of a Differential Equation

Equation (3.16) shows that for the self-adjoint differential operator defined in (3.15) the conjunct is

$$J(v,u) = p(x)(vu'-uv'). \tag{3.18}$$

The expression in parentheses, $vu'-uv'$, is called the Wronskian of v and u and plays a very important role in the theory of differential equations. We shall denote the Wronskian of v and u by the symbol $[v,u]$. The following properties of the Wronskain are evident:

(a) $[v,u] = -[u,v]$;

(b) for any constants α_1 and α_2, we have

$$[\alpha_1 v_1+\alpha_2 v_2,u] = \alpha_1[v_1,u] + \alpha_2[v_2,u] ;$$

(c) for any three functions u(x), v(x) and w(x) we have

$$u[v,w] + v[w,u] + w[u,v] = 0.$$

The last property follows immediately from the expansion of the following determinant:

$$\begin{vmatrix} u & v & w \\ u & v & w \\ u' & v' & w' \end{vmatrix} = 0$$

The following theorem shows why the Wronskian is important:

Theorem II. If the Wronskian of two functions, u(x), v(x) is identically zero, then the functions are linearly dependent in any interval where they do not vanish; conversely, if the functions are linearly dependent, their Wronskian is identically zero. We prove the converse first. Suppose there exist constants α, β, not both zero, such that

$$\alpha u(x) + \beta v(x) = 0 \tag{3.19}$$

for all x in an interval I. Then, differentiating (3.19) with respect to x, we have

$$\alpha u'(x) + \beta v'(x) = 0 \tag{3.20}$$

for all x in I. Equations (3.19) and (3.20) may be considered as homogeneous equations for the quantities α and β. Since the equations have a non-trivial solution, the determinant of the coefficients must be zero, that is,

$$\begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = [u,v] = 0.$$

This proves that the Wronskian is zero for all x in I.

To prove the first part of the theorem, suppose that

$$[u,v] = u(x)v'(x)-v(x)u'(x) = 0$$

for all x in I. Then, if v(x) and u(x) are not zero in I, we have

$$\frac{v'(x)}{v(x)} = \frac{u'(x)}{u(x)}$$

which can be integrated to give

$$\log v(x) = \log u(x) + \log c$$

or

$$v(x) - cu(x) = 0$$

for all x in I. This shows that v(x) and u(x) are linearly dependent.

From Theorem I and (3.18) it follows that if u_1 and u_2 are any solutions of the equation

$$L\,u = 0$$

where L is defined by (3.15), then

$$[u_1,u_2] = u_1u_2' - u_2u_1' = \text{const.}\ /p(x) \tag{3.21}$$

for $0 \leq x \leq 1$. The value of the constant will depend upon the particular solutions $u_1(x)$, $u_2(x)$ that are considered.

Notice that in (3.21) we have implicitly assumed that $p(x) \neq 0$ for $0 \leq x \leq 1$. A point x_o such that $p(x_o) = 0$ is called a <u>singular point</u> of the differential equation. Let us re-write (3.21) in the following form:

$$p(x)(u_1u_2' - u_2u_1') = \text{const.} \tag{3.22}$$

The value of the constant can be determined for any value of $x \neq x_o$. If u_1 and u_2 are not linearly dependent, the value of this constant will not be zero unless $p(x) \equiv 0$. What happens to (3.22) as x approaches x_o? Since p(x) approaches zero, while the product remains a non-zero constant, we conclude that at least one of the functions $u_1(x)$ and $u_2(x)$ must become infinite as x approaches x_o. Later we shall discuss in great detail the nature of this singular behavior of the solutions of the differential equation.

The result (3.21) has been derived only for solutions of a second-order differential equation. There is an analogous result for solutions of an n-th order differential equation or for solutions of a system of differential equations. Before stating the result, we must first extend the definition of the Wronskain so that it will apply to a set of n functions. The Wronskian

of $u_1(x)$, $u_2(x)$, ..., $u_n(x)$ is

$$(3.23)\quad W(u_1,u_2,\ldots u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

The primes and the superscripts denote differentiation with respect to x.

We now state

Theorem III. If $u_1,u_2 \ldots,u_n$ are arbitrary solutions of the differential equation

$$u^{(n)} + p_1 u^{(n-1)} + \ldots + p_n u = 0,$$

then

$$W(u_1,u_2,\ldots,u_n) = c\, e^{-\int p_1 dx},$$

where c is a constant depending on the particular solutions $u_1,u_2,\ldots,u_n$.

This theorem will be an immediate consequence of

Theorem IV. If U_1, U_2, ... U_n are arbitrary vector solutions of the system of n differential equations

$$(3.24)\quad \frac{dU}{dx} = AU$$

and if we define the Wronskian W as the value of the determinant whose columns are the vectors U_1, U_2 ..., U_n, then

$$W = c e^{+\int \text{trace } A dx},$$

where c is a constant whose value depends on the particular solutions $U_1,U_2 \ldots U_n$.

In order to avoid the difficulties of a cumbersome notation we shall prove this theorem for the case n = 2. The general case can be treated in a similar way.

Suppose that

$$U_1 = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$$

and that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

then

$$W = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} .$$

We have

$$\frac{dW}{dx} = \begin{vmatrix} v_1' & v_2' \\ w_1 & w_2 \end{vmatrix} + \begin{vmatrix} v_1 & v_2 \\ w_1' & w_2' \end{vmatrix} . \tag{3.25}$$

Using (3.24) we obtain

$$\begin{pmatrix} v_1' \\ w_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$$

and a corresponding result for v_2, w_2. When these results are used in (3.25) we get

$$\frac{dW}{dx} = \begin{vmatrix} av_1+bw_1 & av_2+bw_2 \\ w_1 & w_2 \end{vmatrix} + \begin{vmatrix} v_1 & v_2 \\ cv_1+dw_1 & cv_2+dw_2 \end{vmatrix}$$

$$= (a+d) \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = W \text{ trace } A,$$

and then an integration gives the conclusion of Theorem IV:

$$W = C\, e^{\int \text{trace} A dx} .$$

Theorem III follows from Theorem IV if A is defined as in (3.5).

There is an interesting consequence of Theorems III and IV which we state as a Corollary. Suppose that in Theorem IV trace A is integrable over the interval (a,b); then if the Wronskian vanishes at one point of the interval it vanishes identically in the whole interval and the vectors $U_1, \cdots, U_n$ are linearly dependent.

The proof is easy. From Theorem IV we know that

$$W = C\, e^{\int \text{tr} A dx} .$$

If W vanishes anywhere, C must be zero since the exponentials never vanish; consequently W vanishes identically. Since W vanishes for all x in (a,b), there exist functions $a_1(x), \cdots, a_n(x)$ such that

$$a_1U_1 + \cdots + a_nU_n = 0 ,$$

for all x in (a,b). At least one of the $a_k(x)$, say $a_1(x)$, is not identically zero; therefore we may write

$$U_1 = b_2U_2 + \cdots + b_nU_n,$$

where $b_k = -a_k/a_1$, $(k = 2,\cdots,n)$. Substituting this expression for U_1 into (3.24) we get

$$U_2 \frac{db_2}{dx} + \cdots + U_n \frac{db_n}{dx} = 0.$$

Either all the derivatives are zero, in which case $b_2,\cdots b_n$ are constants and so U_1 is linearly dependent on $U_2,\cdots,U_n$; or at least one of them, say $\frac{db_2}{dx}$, is not zero, in which case we may write

$$U_2 = + c_3U_3 + \cdots + c_nU_n,$$

where $c_k = -\frac{db_k}{dx} / \frac{db_2}{dx}$, $(k = 3,\cdots,n)$. Now we proceed as above and we show that either $c_3,\cdots,c_n$ are constants, in which case U_2 is linearly dependent on $U_3,\cdots,U_n$, or U_3 can be expressed in terms of $U_4,\cdots,U_n$. It is clear that this reduction can be continued until we prove the corollary.

Problems:

3.7 Prove that Bessel's differential operator defined by

$$Lu = -\frac{1}{x}(xu')' + \frac{\nu^2}{x^2}$$

is positive definite if the boundary conditions are $u(0) = 0$ and $u'(1) + hu(1) = 0$, where h is any positive constant. Hint: Use $w(x) = x$ as a weight function.

3.8. Prove that the operator defined in (3.15) is self-adjoint if the boundary conditions are unmixed or if the boundary conditions are periodic.

3.9. Verify Theorem I if $Lu = u^{(iv)} + k^2u$.

3.10. Verify Theorem I if $\mathcal{S}$ is the space of two-component vector functions U, and if

$$LU = \frac{dU}{dx} - \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U,$$

where a, b, c are constants.

3.11. Prove that

$$[u, [v,w]] + [v, [w,u]] + [w, [u,v]] = 0.$$

3.12 Consider the functions $u(x) = x^2$, $v(x) = x^2 \operatorname{sgn} x$ over the interval $(-1,1)$. The Wronskian of u and v is identically zero, yet u and v are not linearly dependent. Explain the apparent contradiction to Theorem II. Hint: u and v vanish when $x = 0$.

3.13 Suppose that $Lu = -(xu')'$. What is the Wronskian of any two solutions of $Lu = 0$? Investigate the behavior of the Wronskian as x approaches zero.

3.14 Prove Theorem III directly. Hint: Use a method similar to that used in the proof of Theorem IV.

Green's Function

The inverse of an operator L is defined to be operator L^{-1} such that

$$L^{-1}L = I,$$

I being the identity operator. In E_n, I is represented by a matrix with elements δ_{ij} which are defined as

$$\delta_{ij} = 0, \quad i \neq j$$
$$= 1, \quad i = j.$$

In $\mathcal{S}$, I will be represented not by a matrix, but by an integral operator whose kernel is a symbolic function, namely $\delta(x-t)$, which equals zero when $x \neq t$ and which is undefined in value when $x = t$. Therefore, in the case of linear differential operators, the inverse will be found by solving the differential equation

$$Lg = \delta(x-t), \tag{3.26}$$

subject to the given boundary conditions. Here, g is a symbolic function for the distribution in x and t which satisfies (3.26). It will be called the Green's function of the differential operator L in the domain defined by the given boundary conditions.

Since $\delta(x-t)$ is zero in the neighborhood of every value of x except $x = t$, (3.26) implies that $g(x,t)$ satisfies the homogeneous equation

$$Lg = 0 \tag{3.27}$$

except when $x = t$. In the neighborhood of $x = t$, the behavior of $g(x,t)$ will depend on the explicit form of L.

Suppose L is a second-order self-adjoint operator:

$$L = -\frac{d}{dx}\left(p \frac{d}{dx}\right) + q(x), \tag{3.15}$$

and assume $p(t) \neq 0$ in (3.26). Put

$$g(x,t) = -(x-t)H(x-t)/p(t) + k(x,t). \tag{3.28}$$

Now,

$$\frac{d}{dx}\left[(x-t)H(x-t)\right] = H(x-t)$$

and

$$\frac{d}{dx}H(x-t) = \delta(x-t)\ ;$$

consequently from (3.28) and (3.15), we have

(3.29) $$Lk = q(x)(x-t)H(x-t)/p(t)\ .$$

Note that the right side of (3.29) is a continuous function of x. It can be shown that k(x,t) will be a twice differentiable function for all values of x such that $p(x) \neq 0$. In fact, we shall prove the following theorem in Appendix II:

Theorem II-I. Let p(x), q(x), and f(x) be piecewise continuous functions of x in the closed interval (0,1), and let p(x) be positive in that interval; then there exists a continuous function u(x) such that pu' exists and is continuous for all x; furthermore

(II.I) $$(pu') - q(u) = f(x)$$

for all values of x for which both sides are continuous functions of x; and $u(0) = u'(0) = 0$.

Since k(x,t) is twice differentiable, equation (3.28) enables us to state the following theorem about the behavior of the Green's function:

Theorem V: For all values of x, except x = t, the Green's function satisfies the homogeneous equation. In the case where L is given by (3.15), the Green's function at x = t is continuous while its derivative has a jump of magnitude $-1/p(t)$, where $p(t) \neq 0$.

Once the Green's function is known, the non-homogeneous equation

(3.30) $$Lu = f(x)$$

with the assigned boundary conditions can be solved. The solution is

(3.31) $$u(x) = \int_0^1 f(t)g(x,t)\,dt.$$

The proof that u as defined in (3.31) satisfies (3.30) is as follows:

$$Lu = \int_0^1 f(t)\,Lg\,dt = \int_0^1 f(t)\delta(x-t)\,dt = f(x).$$

u also satisfies the assigned boundary conditions because g(x,t) as a function of x satisfies them.

Example One - Green's Function

Let

$$L = -\frac{d^2}{dx^2}$$

and let the boundary conditions be

$$u(0) = u'(0) = 0.$$

To find the Green's function $g(x,t)$ we must solve

$$\frac{d^2g}{dx^2} = -\delta(x-t) \tag{3.32}$$

with the conditions

$$g(0) = g_x(0) = 0. \tag{3.33}$$

As remarked previously, g is a solution of the homogeneous equation

$$\frac{d^2g}{dx^2} = 0, \tag{3.34}$$

except at $x = t$, where it is continuous and its derivative has a jump of magnitude -1. For $x < t$, (3.33) and (3.34) imply that

$$g = 0,$$

while for $x > t$ we assume that g is an arbitrary solution of (3.34), namely

$$g = \alpha x + \beta.$$

Since the jump in the derivative of g is -1, we must have

$$\alpha = -1;$$

since g is continuous at $x = t$, we must have

$$\beta = t.$$

Finally

$$g = 0, \quad x < t$$
$$= t-x, \quad x > t$$

or

$$g(x,t) = -(x-t)H(x-t) \tag{3.35}$$

for all values of x. It is obvious that $g(x,t)$ satisfies (3.32) and (3.33).

Now, the solution of

$$(3.36) \qquad -\frac{d^2u}{dx^2} = f(x)$$

with the conditions

$$(3.37) \qquad u(0) = u'(0) = 0$$

may be found. Using (3.31) and (3.35), we have

$$(3.38) \qquad u(x) = -\int_0^1 f(t)(x-t)H(x-t)dt$$

$$= -\int_0^x f(t)(x-t)dt,$$

and this is clearly the solution of (3.36) satisfying (3.37).

Suppose that, instead of (3.17), the conditions are

$$u(0) = \beta, \quad u'(0) = \alpha.$$

Then, put

$$u = \alpha x + \beta + v;$$

we find that

$$-\frac{d^2v}{dx^2} = f(x),$$

where

$$v(0) = v'(0) = 0.$$

Solving this by means of (3.38) we find that

$$u(x) = \alpha x + \beta - \int_0^x f(t)(x-t)dt.$$

<u>Example Two - Green's Function</u>

Let L again be $-\frac{d^2}{dx^2}$, but now suppose the conditions are

$$(3.39) \qquad u(0) = 0, \quad u(1) = 0.$$

To find the Green's function we must solve

$$\frac{d^2g}{dx^2} = -\delta(x-t)$$

on the manifold defined by the boundary conditions

$$g(0,t) = g(1,t) = 0.$$

We shall illustrate a very useful technique for finding the Green's function. For $x < t$, the solution of the homogeneous equation that satisfies the left boundary condition $g(0,t) = 0$ will be proportional to x. For $x > t$, the solution of the homogeneous equation that satisfies the right boundary condition, $g(1,t) = 0$, will be proportional to $1 - x$.

Write

$$\begin{aligned} g(x,t) &= x \quad , \qquad x < t \\ &= 1-x, \qquad x > t. \end{aligned}$$

This is wrong because $g(x,t)$ is not continuous for $x = t$. The value of g as x approaches t from below is t, while the value of g as x approaches t from above is $1 - t$. Multiply the first expression for g by the value of the second expression at $x = t$ and multiply the second expression for g by the value of the first expression at $x = t$. Then write

$$\begin{aligned} (3.40) \qquad g(x,t) &= x(1-t), \qquad x < t \\ &= (1-x)t, \qquad x > t. \end{aligned}$$

This function is continuous at $x = t$. Because the derivative of this function for $x < t$ is $(1-t)$ while the derivative for $x > t$ is $-t$, the jump in the derivative at $x = t$ is

$$-t - (1-t) = -1 .$$

(3.40) is therefore the Green's function for the operator $-\frac{d^2}{dx^2}$ in the manifold defined by the conditions (3.39). We may write the expression for $g(x,t)$ as follows:

$$\begin{aligned} g(x,t) &= x(1-t)H(t-x) + (1-x)tH(x-t) \\ &= \frac{x+t}{2} - xt - \frac{|x-t|}{2} . \end{aligned}$$

Note that $g(x,t)$ is symmetric in x and t. We shall see later that this is generally true for the Green's function of a self-adjoint operator.

By differentiation we may check that (3.40) is actually the solution of the differential equation. We have

$$\frac{dg}{dx} = \frac{1}{2} - t - \frac{1}{2}\,\mathrm{sgn}(x-t)$$

$$\frac{d^2g}{dx^2} = -\,\delta(x-t).$$

As in the preceding example we find that the solution of

$$-\frac{d^2}{dx^2}\,u = f(x)$$

with the boundary conditions

$$u(0) = u(1) = 0$$

is

$$u = \int_0^1 f(t)\left[x(1-t)H(t-x) + (1-x)t\,H(x-t)\right]dt$$

$$= (1-x)\int_0^x t\,f(t)dt + x\int_x^1 (1-t)f(t)dt.$$

Example Three - Green's Function

Consider the operator

$$L = -\frac{d^2}{dx^2} - 4$$

in the manifold defined by

$$u(0) = u'(1) = 0.$$

The Green's function is found by solving the equation

$$\frac{d^2g}{dx^2} + 4g = -\,\delta(x-t), \tag{3.41}$$

where

$$g(0,t) = g_x(1,t) = 0. \tag{3.42}$$

We use the same technique as in Example Two. The solutions of the homogeneous equation corresponding to (3.41) are any linear combination of sin 2x and cos 2x. We pick two solutions of the homogeneous equation - one to satisfy the boundary condition at zero, the other to satisfy the boundary condition at unity. We may then write

$$g(x,t) = \sin 2x \qquad x < t$$
$$= \cos 2(1-x), \qquad x > t.$$

Since $g(x,t)$ is not continuous for $x = t$, we multiply the first expression by

the value of the second expression at $x = t$ and also multiply the second expression by the value of the first at $x = t$. We get

$$g(x,t) = \sin 2x \cos 2(1-t), \quad x < t$$
$$= \cos 2(1-x) \sin 2t, \quad x > t.$$

Now, $g(x,t)$ is continuous, but the jump in the magnitude of the derivative at $x = t$ is

$$2 \sin 2(1-t) \sin 2t - 2\cos 2t \cos 2(1-t)$$
$$= -2 \cos 2(t+1-t) = -2 \cos 2.$$

Since it follows from (3.41) that the jump in the derivation should be -1, we see that this last formula for $g(x,t)$ is still wrong. To correct it, we must divide it by $2 \cos 2$. We have then the final and correct results:

$$g(x,t) = \sin 2x \cos 2(1-t)/2 \cos 2, \quad x < t$$
$$= \cos 2 (1-x) \sin 2t/2 \cos 2, \quad x > t$$

or

$$g(x,t) = \frac{\sin 2x \cos 2(1-t)}{2 \cos 2} H(t-x) + \frac{\cos 2(1-x) \sin 2t}{2 \cos 2} H(x-t). \tag{3.43}$$

Note again that $g(x,t)$ is symmetric in x and t.

It is easy to show by straightforward differentiation that $g(x,t)$ as given by (3.43) satisfies (3.41). We have

$$\frac{dg}{dx} = \frac{2 \cos 2x \cos 2(1-t)}{2 \cos 2} H(t-x) + \frac{2 \sin 2(1-x) \sin 2t}{2 \cos 2} H(x-t)$$
$$- \frac{\sin 2x \cos 2(1-t)}{2 \cos 2} \delta(t-x) + \frac{\cos 2(1-x) \sin 2t}{2 \cos 2} \delta(x-t).$$

The sum of the last two terms is zero, because

$$f(x)\delta(x-t) = f(t)\delta(x-t)$$

and

$$\frac{\sin 2x \cos 2(1-t)}{2 \cos 2}\Bigg|_{x=t} = \frac{\cos 2(1-x) \sin 2t}{2 \cos 2}\Bigg|_{x=t}.$$

Differentiating the formula for $\frac{dg}{dx}$, we find that

$$\frac{d^2g}{dx^2} = -4\,\frac{\sin 2x \cos 2(1-t)}{2\cos 2}\,H(t-x) - 4\,\frac{\cos 2(1-x)\sin 2t}{2\cos 2}\,H(x-t)$$

$$- \frac{\cos 2x \cos 2(1-t)}{\cos 2}\,\delta(t-x) + \frac{\sin 2(1-x)\sin 2t}{\cos 2}\,\delta(x-t)$$

$$= -4g + \frac{\delta(x-t)}{\cos 2}\left[\sin 2(1-x)\sin 2t - \cos 2x \cos 2(1-t)\right]$$

$$= -4g - \delta(x-t),$$

which proves that $g(x,t)$ satisfies (3.41).

Green's Function in the Case of Unmixed Boundary Conditions

The technique that has been used in the two previous examples can be applied to any second-order differential operator if the boundary conditions are not mixed, that is, if each boundary condtion involves just one point of the boundary but not both. A condition such as $u'(0) = u(0)$ is not mixed but a condition such as $u(1) = u(0)$ is mixed.

We shall obtain the Green's function for the general self-adjoint operator

$$L = -\frac{d}{dx}\left(p\,\frac{d}{dx}\right) + q$$

when the domain is defined by the general unmixed boundary conditions

$$B_1(u) = B_2(u) = 0. \tag{3.44}$$

We shall assume $p(x) \neq 0$ in the interval $(0,1)$ and that $B_1(u)$ involves values of u and its derivatives at $x = 0$ only, while $B_2(u)$ involves values of u and its derivatives at $x = 1$ only.

The Green's function is the solution of

$$(pg_x)_x - qg(x,t) = -\delta(x-t) \tag{3.45}$$

with the boundary conditions

$$B_1(g) = B_2(g) = 0. \tag{3.46}$$

Consider the homogeneous equation

$$(pu_x)_x - qu = 0 \tag{3.47}$$

without any boundary condtions. Suppose that $v_1(x)$ and $v_2(x)$ are any two independent solutions of (3.47). Let $w_1(x)$ be a linear combination of $v_1(x)$ and $v_2(x)$ which satisfies the condition

$$B_1(w_1) = 0.$$

Similarly, let $w_2(x)$ be a linear combination of $v_1(x)$ and $v_2(x)$ which satisfied the condition

$$B_2(w_2) = 0 .$$

Then we start the construction of the Green's function by writing

$$g(x,t) = w_1(x), \quad x < t$$
$$= w_2(x), \quad x > t .$$

This expression for $g(x,t)$ will satisfy the differential equation (3.45) for $x \neq t$ and also the boundary conditions (3.46), but it will not satisfy the continuity or the jump conditions at $x = t$. To make $g(x,t)$ continuous, multiply the first expression by the value of the second at $x = t$ and multiply the second expression by the value of the first at $x = t$ so that we have

$$g(x,t) = w_1(x)w_2(t), \quad x < t \tag{3.48}$$
$$= w_2(x)w_1(t), \quad x > t .$$

This expression still does not satisfy the jump condition in the derivative at $x = t$. The jump in the derivative at $x = t$ is given by

$$w_2'(t)w_1(t) - w_1'(t)w_2(t) = \left[w_1,w_2\right]_{x = t},$$

while it should be $-1/p(t)$. This can be remedied by dividing (3.48) by the conjunct of w_2, w_1. The final and correct formula is then

$$g(x,t) = w_1(x)w_2(t)/\left[w_2,w_1\right]p(t), \quad x < t \tag{3.49}$$
$$- w_2(x)w_1(t)/\left[w_2,w_1\right]p(t), \quad x > t,$$

or, written in another way,

$$g(x,t) = \frac{w_1(x)w_2(t)H(t-x) + w_2(x)w_1(t)H(x-t)}{\left[w_2,w_1\right]\, p(t)} . \tag{3.50}$$

The conjunct in the denominator of these expressions is to be evaluated at $x = t$. However, as we have shown in (3.22),

$$p(x)\left[w_2,w_1\right] = \text{constant};$$

consequently instead of evaluating the conjunct at $x = t$, we may evaluate it at any convenient point. For a simple illustration of the usefulness of this fact, consider Example Three of the Green's function. There, the conjunct of $\cos 2(1-x)$ and $\sin 2x$ had to be found. Now, since $p(x) = 1$, we have

$$[\cos 2(1-x), \sin 2x] = \text{constant}$$
$$= 2 \cos 2(1-x) \cos 2x + 2 \sin 2(1-x) \sin 2x$$
$$= 2 \cos 2$$

when we put $x = 0$. Before, this result was obtained by using the addition theorem for the cosine, but now it is an immediate consequence of the properties of the conjunct.

Formulas (3.49) and (3.50) break down if $[w_2,w_1] = 0$. The fact that the conjunct is zero implies that one function, say $w_2(x)$, is a multiple of the other, $w_1(x)$. Since $B_1(w_1) = 0$ and $B_2(w_2) = 0$, it follows that $B_1(w_2) = B_2(w_2) = 0$. This means $w_2(x)$ is non-trivial solution of the homogeneous equation

$$(pu')' - qu = 0$$

with the boundary conditions

(3.44) $$B_1(u) = B_2(u) = 0.$$

Consequently, $w_2(x)$ is an eigenfunction of the operator L with the boundary conditions (3.44), and the corresponding eigenvalue is $\lambda = 0$.

The preceding discussion may be summarized in the following

Theorem VI. Let $w_1(x)$ be the solution of

$$Lw = -(pw')' + qw = 0$$

satisfying the unmixed boundary condition $B_1(w) = 0$, and let $w_2(x)$ be the solution of the same differential equation satisfying the unmixed boundary condition $B_2(w) = 0$; then the Green's function of L with the boundary conditions B_1 and B_2 is given by the following formula:

$$g(x,t) = \frac{w_1(x)w_2(t)H(t-x) + w_2(x)w_1(t)H(x-t)}{J(w_2,w_1)}$$

where $J(w_2,w_1)$ is the conjunct of w_2 and w_1 evaluated at any point.

Problems:

3.15 Find the Green's function for $L = -\frac{d^2}{dx^2}$ in the manifold defined by the boundary conditions $u'(0) = 0$, $u(1) = 0$.

3.16 Find the Green's function for $L = -\frac{d^2}{dx^2} - k^2$, k constant, in the manifold defined by the boundary conditions $u(0) = 0$, $u(1) = 0$. For what values of k does the formula break down?

3.17 Find the Green's function for $L = -\frac{d}{dx}\left(x\frac{d}{dx}\right)$ in the manifold defined by the boundary conditions $u(0) = 0$, $u'(1) = 0$.

3.18 Use the Green's function to find a function $u(x)$ such that $u'' + k^2u = f(x)$ and such that $u'(0) = u'(1) = 0$.

3.19 Show that formula (3.50) satisfies the conditions of Theorem V.

The Case where the Homogeneous Equation has a Non-Trivial Solution

Suppose there exists a non-zero solution $w(x)$ of the homogeneous equation

$$-Lw = (pw_x)_x - qw = 0$$

such that

$$B_1(w) = B_2(w) = 0.$$

Consider now the problem of finding $u(x)$ such that

$$(pu_x)_x - qu = f(x)$$

with the conditions

$$B_1(u) = B_2(u) = 0.$$

It can be proved that if L is self-adjoint and has a closed range and if the homogeneous equation

$$Lu = 0$$

has a single non-trivial solution w, then the non-homogeneous equation

$$Lu = f$$

has a solution if and only if

$$(f,w) = 0.$$

We shall show that a similar theorem is valid for differential operators.

Theorem VII. Let $w(x)$ be the solution, unique except for a constant factor, of the homogeneous differential equation

$$Lw = -(pw')' + qw = 0$$

such that $B_1(w) = B_2(w) = 0$; then the non-homogeneous equation

(3.51) $$Lu = f$$

has a solution satisfying the boundary conditions $B_1(u) = B_2(u) = 0$ if and only if

(3.52) $$\int_0^1 f(x)w(x)dx + 0.$$

We shall here assume that the boundary conditions are unmixed. The theorem is valid for arbitrary boundary conditions.

If a function u exists satisfying (3.51) and the boundary conditions, then since L is self-adjoint, we have

$$0 = (u,Lw) - (Lu,w) = -(f,w)$$

and this shows that (3.52) is satisfied. Conversely, if we suppose that (3.52)

is satisfied, then we may solve (3.51). Let $v(x)$ be a solution of

$$(pu_x)_x - qu = 0$$

and let $v(x)$ be independent of $w(x)$. Then, by using the Green's function technique as illustrated in Example One, we find that a solution of (3.51) which satisfies the conditions $u(0) = u'(0) = 0$, instead of the conditions $B_1(u) = B_2(u) = 0$, is

$$u(x) = \frac{w(x)}{p[w,v]} \int_0^x fv\,d\tau - \frac{v(x)}{p[w,v]} \int_0^x fw\,d\tau \quad . \tag{3.53}$$

Note that, as shown previously, $p[w,v]$ is constant for all values of x.

We can now show that the function defined by (3.53) will be the solution of the original problem. It is clear that (3.53) satisfies the differential equation (3.51). We now investigate the boundary conditions. Since the boundary conditions are linear, homogeneous, and unmixed, we may write them as follows:

$$B_1(u) = \alpha_1 u(0) + \beta_1 u'(0)$$
$$B_2(u) = \alpha_2 u(1) + \beta_2 u'(1),$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are given constants. Since $u(x)$ in (3.53) was constructed so that $u(0) = u'(0) = 0$, it is obvious that $B_1(u) = 0$. From (3.53) we have

$$u(1) = \frac{w(1)}{p[w,v]} \int_0^1 fv\,d\tau \quad , \tag{3.54}$$

because from (3.50),

$$\int_0^1 fw\,d\tau = 0.$$

If we differentiate (3.53) using condition (3.52) and put x equal to unity, we obtain

$$u'(1) = \frac{w'(1)}{p[w,v]} \int_0^1 fv\,d\tau. \tag{3.55}$$

Formulas (3.54) and (3.55) together show that

$$B_2(u) = \frac{B_2(w)}{p[w,v]} \int_0^1 fv\,d\tau = 0$$

because of the assumption on w. We have therefore proved Theorem VII and we have also shown that if (3.52) is satisfied then the solution of (3.51) which

satisfies the conditions

$$u(0) = u'(0) = 0$$

will also satisfy the boundary conditons

$$B_1(u) = B_2(u) = 0.$$

We could not use the Green's function technique in this case because the equation

(3.56) $$(pu_x)x - qu = -\delta(x-t)$$

with the boundary conditions $B_1(u) = B_2(u) = 0$ has no solution, since

$$\int \delta(x-t)w(x)dx \neq 0,$$

that is, (3.53) is not satisfied. If the right side of (3.56) is modified so that it is orthogonal to $w(x)$, we obtain a problem which can be solved. This is done by writing

(3.57) $$(pu_x)_x - qu = -\delta(x-t) + \gamma w(x),$$

where γ is found from the equation

$$w(t) - \gamma\int_0^1 w^2dx = 0.$$

(3.57) can be solved in the same way as (3.51) was. The details will be left to the reader.

Problem:

3.20. In each of the following cases find the appropriate orthogonality relation that $f(x)$ must satisfy in order that the equation may have a solution, and then find the solution:

(a) $u'' = f(x)$, $u'(0) = u'(1) = 0$;

(b) $u'' + \pi^2 u = f(x)$, $u(0) = u(1) = 0$;

(c) $(xu')' = f(x)$, $xu'(x)\Big|_{x=0} = 0$, $u'(1) = 0$;

(d) $u'' = f(x)$, $u(0) = u(1)$, $u'(0) = u'(1)$.

Green's Function for General Boundary Conditions

In the general case where the boundary conditions are mixed, the Green's function may still be found in a straightforward way. Suppose the equation is

$$(3.58) \qquad (pg_x)_x - qg = -\delta(x-t)$$

with the boundary conditions

$$(3.59) \qquad B_1(g) = B_2(g) = 0.$$

Let $v(x)$ and $w(x)$ be any two linearly independent solutions of

$$(pu_x)_x - qu = 0.$$

Then write

$$(3.60) \qquad g(x,t) = \alpha_1 v(x) + \beta_1 w(x) + \frac{v(x)w(t)H(t-x)+w(x)v(t)H(x-t)}{[w,v]\ p(t)},$$

where α_1 and β_1 are constants which will be determined so that $g(x,t)$ satisfies the boundary conditions (3.59). The bracket expression $[w,v]$ is the Wronskian of w and v at the value t. Note that it cannot be zero for any value of t because if it were zero, then by the Corollary to Theorem IV and by Theorem II, the functions $v(x)$ and $w(x)$ would not be linearly independent.

We show first that $g(x,t)$ satisfies the appropriate conditions at $x = t$, namely that $g(x,t)$ is continuous and its derivative has a jump of magnitude $-1/p(t)$. The continuity is obvious. From the definition of the Wronskian it follows that the jump in the derivative at $x = t$ is

$$\frac{w'(t)v(t) - v'(t)w(t)}{[w,v]p(t)} = -\frac{1}{p(t)} .$$

Now consider the boundary conditions. Since they are linear and homogeneous, we have

$$(3.61) \qquad B_1(g) = \alpha_1 B_1(v) + \beta_1 B_1(w) + B_1(r)$$

$$B_2(g) = \alpha_1 B_2(v) + \beta_1 B_2(w) + B_2(r),$$

where we have used $r(x)$ to represent the function

$$\frac{v(x)w(t)H(t-x) + w(x)v(t)H(x-t)}{[w,v]p(t)} .$$

The boundary conditions (3.59) will consequently give us two linear equations to determine the values of α_1 and β_1. These equations will have a solution if the determinant

$$(3.62) \qquad \begin{vmatrix} B_1(v) & B_1(w) \\ B_2(v) & B_2(w) \end{vmatrix}$$

does not vanish.

If (3.62) vanishes, then either $B_1(v) = B_2(v) = 0$, which implies that v is an eigenfunction of the operator corresponding to the eigenvalue zero, or there exists a constant c such that

$$B_1(w) + c\, B_1(v) = 0$$

$$B_2(w) + c\, B_2(v) = 0.$$

These equations imply that $B_1(w + cv) = B_2(w + cv) = 0$; hence $w + cv$ is an eigenfunction of the operator and corresponds to the eigenvalue zero.

We see then that <u>if there is no eigenfunction corresponding to the eigenvalue zero, the Green's function exists and is given by (3.60) where α and β are found by solving (3.59).</u>

<u>Green's Function for an n-th Order Equation or a System</u>

The method of the preceding section can be easily generalized in order to obtain the Green's function for an n-th order differential equation or for a system of n first-order differential equations.

The Green's function for an n-th order differential operator is defined to be the symbolic function $g(x,t)$ which satisfies the equation

$$(3.63) \qquad Lg = p_0 \frac{d^n g}{dx^n} + p_1 \frac{d^{n-1} g}{dx^{n-1}} + \ldots + p_{n-1} \frac{dg}{dx} + p_n g = -\delta(x-t)$$

and the boundary conditions

$$B_1(g) = B_2(g) = \ldots = B_n(g) = 0.$$

Since the δ-function is zero except when $x = t$, equation (3.63) implies that g satisfies the homogeneous equation

$$Lg = 0$$

for all values of x other than $x = t$. If we integrate (3.36) with respect to x from 0 to x we get

$$p_o \frac{d^{n-1}g}{dx^{n-1}}\Big|_o^x - \int_o^x p_o' \frac{d^{n-1}g}{dx^{n-1}}\,dx + \int_o^x \left[p_1 \frac{d^{n-1}g}{dx^{n-1}} + \ldots + p_n g \right] dx = -H(x-t).$$

If we assume that the functions $p_k (k = 0, 1..n)$ have continuous derivatives of order n-k then this result implies that $p_o \frac{d^{n-1}g}{dx^{n-1}}$ has a jump of magnitude (-1) at $x = t$. Further integrations show that the functions $p_o \frac{d^{n-2}g}{dx^{n-2}}$, $p_o \frac{d^{n-3}g}{dx^{n-3}}$, ... $p_o g$ are continuous at $x = t$.

Assume that $p_o(x)$ is continuous and non-zero in the interval (0,1); then we conclude that if g satisfies (3.63) it will also satisfy the following conditions: g(x,t), together with its first n-2 derivatives, will be continuous at $x = t$, while the (n-1)th derivative will have a jump of magnitude $-1/p_o(t)$ there. This conclusion can be easily verified. Put

$$(3.64) \qquad g(x,t) = -\frac{(x-t)^{n-1}}{(n-1)!\, p_o(t)} H(x-t) + k(x,t);$$

then g will satisfy the above conditions if k(x,t) and its first (n-1) derivatives are continuous at $x = t$. Now if (3.64) is substituted in (3.63) we find that

$$(3.65) \qquad Lk = r(x,t),$$

where r(x,t) is a piecewise continuous function of x. Just as in Appendix II, we may prove that equation (3.65) has a continuous solution such that the first n-1 derivatives of this solution is likewise continuous.

Now since the solution of (3.63) which satisfies the boundary conditions is unique, g(x,t) must be of the form (3.64) and therefore must obey the continuity conditions stated above.

The definition of the Green's function for a system is more complicated, and will be considered in Appendix I.

Green's Function and the Boundary Value Problem

So far, we have discussed the solution of non-homogeneous differential equations with homogeneous boundary conditions. We now wish to consider the case of non-homogeneous boundary conditions, both for homogeneous and non-homogeneous differential equations. We shall show that the Green's function technique can be used to solve these more complicated problems also.

For example, suppose that we wish to find a solution u of the differential equation

$$Lu = u'' + a(x)u' = f \tag{3.66}$$

such that $u(0) = b_o$, $u(1) = b_1$. Let $h(x,t)$ be the Green's function of the adjoint homogeneous equation; then

$$L^*h = \frac{d^2h}{dx^2} - \frac{d}{dx}(ah) = \delta(x-t) \tag{3.67}$$

and $h(x,t)$ satisfies the homogeneous adjoint boundary conditions

$$h(0,t) = h(1,t) = 0.$$

Multiply equation (3.66) by h, equation (3.67) by u, subtract, and integrate over (0,1). We get

$$\int_0^1 u(x)\delta(x-t)dx - \int_0^1 h(x,t)f(x)dx = (u,L^*h) - (h,Lu) = (uh'-hu')\Big|_0^1$$

$$= u(1)h'(1,t) - u(0)\,h'(0,t)$$

or, finally,

$$u(t) = \int_0^1 h(x,t)f(x)dx + b_1h'(1,t) - b_oh'(0,t). \tag{3.68}$$

The primes denote differentiation with respect to x.

The same method can be used in more general cases. Before proving this fact, we must investigate a little more closely the boundary conditions of the adjoint problem. From equations (3.9) and (3.10) we have

$$(v,Lu) - (u,L^*v) = J(v,u)\Big|_0^1. \tag{3.69}$$

Suppose that the boundary conditions associated with the operator L are

$$B_1(u) = b_1,\ B_2(u) = b_2, \cdots,\ B_n(u) = b_n.$$

Since the boundary conditions depend linearly on the 2n values of u and the first

(n-1) derivatives of u at x = 0 and x = 1, we may solve for any n of these numbers, say $u(0), u'(0),\ldots, u^{(k)}(0), u(1), u'(1),\ldots, u^{(n-k-2)}$, in terms of $b_1,\ldots, b_n$. When these expressions are substituted into $J(v,u)\Big|_0^1$, we get

$$J(v,u)\Big|_0^1 = b_1C_1^*(v)+\ldots b_nC_n^*(v) + \text{other terms.} \tag{3.70}$$

A simple calculation shows that the expressions $C_1^*(v),\ldots, C_n^*(v)$ are linear homogeneous expressions containing the values of v and its first (n-1) derivatives at x = 0 and x = 1. The "other terms" in (3.70) are linear homogeneous expressions involving the n numbers, $u^{(k)}(0),\ldots, u^{(n-1)}(0)$, $u^{(n-k-1)}(1),\ldots, u^{(n-1)}(1)$. We shall denote these n numbers by $C_1(u),\ldots, C_n(u)$ respectively; then the "other terms" in (3.70) may be written as

$$- C_1(u)\, B_1^*(v) -\ldots- C_n(u)\, B_n^*(v),$$

where $B_1^*(v),\cdots,B_n^*(v)$ are linear homogeneous expressions involving the values of v and its first (n-1) derivatives at x = 0 and x = 1. Finally, instead of (3.70), we have

$$J(u,v)\Big|_0^1 = B_1(u)C_1^*(v)+ \cdots + B_n(u)C_n^*(v) - C_1(u)B_1^*(v)- \cdots-C_n(u)B_n^*(v); \tag{3.71}$$

consequently, the homogeneous adjoint boundary conditions are

$$B_1^*(v) = \cdots = B_n^*(v) = 0.$$

As an illustration, suppose that $Lu = u^{(iv)}$ and that

$B_1(u) = u(0)+u(1)$, $B_2(u) = u'(0)$, $B_3(u) = u''(0) + u'(1)$, $B_4(u) = u''(1)$.

The operator L is formally self-adjoint, and by evaluating (v,Lu) - (u,Lv) we find that

$$J(v,u)\Big|_0^1 = \left[vu''' - v'u''+ v''u' - v'''u\right]_0^1 .$$

Eliminating u(0), u'(0), u'(1) and u''(1) from this formula, we get

$$\begin{aligned} J(v,u)\Big|_0^1 &= v(1)u'''(1)-v'(1)B_4+v''(1)(B_3-u''(0))-v'''(1)u(1) \\ &\quad - v(0)u'''(0)+v'(0)u''(0)-v''(0)B_2+v'''(0)(B_1-u(1)) \\ &= B_1v'''(0)-B_2v''(0) + B_3v''(1) - B_4v'(1) \\ &\quad + u'''(1)v(1)+u''(0)(v'(0)-v''(1))-u'''(0)v(0)-u(1)(v'''(1)+v'''(0)); \end{aligned}$$

consequently,

$$B_1^*(v) = -v(1),\ B_2^*(v) = -v'(0)+v''(1),\ B_3^*(v) = v(0),\ B_4^*(v) = v'''(1)+v'''(0).$$

Let us now return to the boundary value problem. Suppose that the differential equation to be solved is

$$Lu = f \tag{3.72}$$

with the non-homogeneous boundary conditions

$$B_1(u) = b_1, \ldots, B_n(u) = b_n.$$

Let $h(x,t)$ be the Green's function of the adjoint homogeneous equation: thus

$$L^*h = \delta(x-t), \tag{3.73}$$

and $h(x,t)$ satisfies the homogeneous adjoint boundary conditions

$$B_1^*(h) = \ldots = B_n^*(h) = 0.$$

Multiply equation (3.72) by h, equation (3.73) by f, subtract , and integrate over (0,1). Then, using (3.71) we get

$$\int_0^1 u(x)\delta(x-t)dx - \int_0^1 h(x,t)f(x)dx = (u,L^*h) - (h,Lu) = -J(h,u)\Big|_0^1$$

$$= B_1(u)C_1^*(h) + \ldots + B_n(u)C_n^*(h) - C_1(u)B_1^*(h) - \ldots - C_n(u)B_n^*(h).$$

If we make use of the boundary conditions on u and h, this expression reduces to the following:

$$u(t) = \int_0^1 h(x,t)f(x)dx + b_1C_1^*(h) + \ldots + b_nC_n^*(h). \tag{3.74}$$

This proves

Theorem VIII. <u>The solution of $Lu = f$ which satisfies the boundary conditions $B_1(u) = b_1, \ldots, B_n(u) = b_n$ is given by (3.74) where $h(x,t)$ is the Green's function of the adjoint homogeneous equation (3.73).</u>

Note that $C_1^*(h), \ldots, C_n^*(h)$, which appear in (3.74), are actually functions of t. Note also that the method indicated above works just as well for a system of first-order differential equations as for an n-th-order differential equation.

<u>Problems</u>:

3.23 Solve $u'' + u = f(x)$ with the conditions $u'(0) = 0$, $u(0) + u(1) = 2$.

3.24 Solve $u^{(iv)} = f$ with the conditions $B_1(u) = 1$, $B_2(u) = 0$, $B_3(u) = 0$ and $B_4(u) = 1$, where B_1, B_2, B_3 and B_4 are the expressions used in the text in connection with this operator.

3.25 Solve the system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} o & k \\ k & o \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

with the boundary conditions $u_1(o) = 1$ and $u_2(1) = 2$.
(Hint. Refer to Appendix I for the Green's matrix of a solution).

Green's Function of the Adjoint Equation

Suppose that in Theorem VIII we consider the case of homogeneous boundary conditions, that is, $b_1 = \ldots = b_n = 0$; then the solution of $Lu = f$ as given by (3.74) is

$$u(t) = \int_o^1 h(x,t)f(x)dx.$$

However, we know that the solution can also be written as

$$u(x) = \int_o^1 g(x,t)f(t)dt,$$

where $g(x,t)$ is the Green's function of the original operator L. This indicates that there is a connection between $g(x,t)$ and $h(x,t)$. We prove

Theorem IX. Let $g(x,t)$ be the Green's function for an operator L on a manifold defined by certain boundary conditions, and let $h(x,t)$ be the Green's function for the adjoint operator L^* on the manifold defined by the adjoint boundary conditions; then

$$g(x,t) = h(t,x).$$

In particular, if L is self-adjoint then $g(x,t)$ is a symmetric function of x and t.

To prove the theorem we start with the defining relations for the Green's functions, namely

$$Lg(x,t) = \delta(x,-t),$$

(3.75)

$$L^*h(x,\tau) = \delta(x-\tau).$$

From the properties of the adjoint operator, we get

$$\int h(x,\tau)Lg(x,t)dx = \int L^*h(x,\tau)g(x,t)dx.$$

If we use (3.75), this equation becomes

$$\int h(x,\tau)\delta(x-t)dx = \int \delta(x-\tau)g(x,t)dx,$$

or

$$h(t,\tau) = g(\tau,t);$$

this last result, with a slight change of notation, is the conclusion of Theorem IX.

Problem:

3.26 Verify Theorem IX if $L = \frac{d^2}{dx^2} + x\frac{d}{dx}$, with the conditions $u(0) = u(1)$ and $u'(0) + u'(1) = 0$.

Discontinuity Conditions

In many practical applications it is important to consider linear differential equations in which the coefficients may be discontinuous. This situation occurs in cases where the properties of the medium vary discontinuously, for example the propagation of sound waves through media of different densities or the transmission of heat through materials of different thermal conductivity. The differential equation describes the behavior of the solution in each medium separately, but it is still necessary to determine how the solution behaves across the interface of the two media. Usually the behavior of the solution across the interface, from which we derive the so-called discontinuity conditions, is determined by physical considerations; for example, in sound propagation the pressure and velocity of the sound wave must be continuous across the interface, otherwise there would be an infinite acceleration there. Similarly, in electromagnetic theory we can show that because of Gauss' theorem the tangential components of the electric field must be continuous across the interface. Clearly, these discontinuity conditions are as much a part of the physical description of the problem as are the differential equations themselves, and consequently the question of discontinuity conditions is not a mathematical question. However, it is useful to investigate what kind of discontinuity conditions are mathematically appropriate for given differential equations.

Consider the self-adjoint differential equation

$$-Lu = (pu')' - qu = f(x) \tag{3.76}$$

where $p(x)$, $q(x)$, and $f(x)$ are continuous except for possible jumps at the points $x = x_1, x_2, \cdots, x_k$. Theorem II-I tells us that despite the jumps there exist solutions of (3.76) such that the functions $u(x)$ and $p(x)u'(x)$ are continuous throughout the entire interval; consequently we may write the discontinuity conditions for this equation as follows:

$$u(x_i+0) = u(x_i-0)$$

$$p(x_i+0)u'(x_i+0) = p(x_i-0)u'(x_i-0), \quad i = 1,2,\ldots k.$$

Note that if $p(x)$ is continuous at $x = x_i$, the last condition states that $u'(x)$ must be continuous at $x = x_i$.

Instead of using the results of a theorem such as Theorem II-1 about the existence of a solution of the differential equation, we may obtain the discontinuity conditions by the following simple argument. Integrate equation (3.76) across a discontinuity, say from $x = x_i-\epsilon$ to $x = x_i+\epsilon$, where ϵ is a

small positive quantity. We have

$$pu' \Big|_{x_i-\epsilon}^{x_i+\epsilon} = \int_{x_i-\epsilon}^{x_i+\epsilon} (qu + f)dx.$$

Since q, u, and f are integrable functions, the right-hand side of this equation approaches zero as ϵ approaches zero, and therefore pu' is continuous across x_i. Now write (3.76) in an integral form:

$$u' = \frac{p(0)u'(0)}{p(x)} + \frac{1}{p(x)}\int_0^x (qu+f)d\xi \ ;$$

This is valid if $p(x) \neq 0$. If we integrate again across x_i we get

$$u\Big|_{x_i-\epsilon}^{x_i+\epsilon} = \int_{x_i-\epsilon}^{x_i+\epsilon} \frac{p(0)u'(0)}{p(x)}\,dx + \int_{x_i-\epsilon}^{x_i+\epsilon} \frac{dx}{p(x)} \int_0^x (qu+f)d\xi \ .$$

The integrals again approach zero as ϵ approaches zero and therefore u(x) also is continuous at $x = x_i$.

The same conclusions about the continuity of u(x) and pu' may be obtained by a different argument. Suppose u(x) has a jump of magnitude α at $x = x_i$; then we can write

$$u(x) = \alpha H(x-x_i) + v(x),$$

where v(x) is continuous at $x = x_i$. We would now find that Lu is not a piecewise continuous function but instead contains the derivative of a delta function. Similarly, if pu' has a jump discontinuity, Lu contains a delta function. Since Lu is given as a piecewise continuous function, it follows that both u and pu' must be continuous. We state our conclusions in the following

Rule: The discontinuity conditions for the operator $Lu = -(pu') + qu$ are that both u and pu' be continuous.

This rule has an important consequence which is based on the concept of the impedance of a solution. We define the impedance of a solution u(x) of the equation $Lu = -(pu') - qu$ as the ratio $\frac{pu'}{u}$. Then the above rule implies that the impedance is continuous across a discontinuity.

As an application of these rules we discuss the propagation of a plane wave through two media of different properties. The mathematical problem that corresponds to this physical question is the following:

Consider the equation

$$u'' + k^2 u = 0, \quad -\infty < x < \infty$$

where k has one constant value, k_1, for $x < x_o$ and has another constant value, k_2, for $x > x_o$. Find a solution u of this equation such that, for x very large negatively, $u = e^{-ik_1 x}$.

For $x < x_o$, $e^{ik_1 x}$ is a solution of the differential equation. For $x > x_o$, $Ae^{ik_2(x+x_o)} + Be^{ik_2(x-x_o)}$ is the form of the general solution of the equation. Since the properties of the medium change discontinuously across $x = x_o$, then by the rule stated above, both u and u' (note p is identically unity) must be continuous at $x = x_o$. These conditions give the following equation for A and B:

$$\exp(ik_1 k_o) = A + B$$

$$ik_1 \exp(-ik_1 x_o) = ik_2(A - B).$$

By division we find that

$$k_2 \frac{A - B}{A + B} = k_1 ,$$

and then

$$\frac{A}{B} = \frac{k_2 + k_1}{k_2 - k_1} .$$

The fact that this result could have been obtained directly, by using only the impedance rule, illustrates the advantages of this rule. It enables us to obtain the ratio A/B immediately without solving a set of linear equations. In more complicated cases where there is more than one discontinuity present, such a use of the impedance rule may produce a great simplification.

Of course, to find the values of A and B we can solve the original equations. The solution is

$$A = \frac{k_1+k_2}{2k_2} \exp(ik_1 x_o), \quad B = \frac{k_2-k_1}{2k_2} \exp(-ik_1 x_o).$$

The physical interpretation of the solution is that a plane wave $B\exp[-ik_2(x-x_o]$ comes in from the right; one part, namely $A \exp[ik_2(x-x_o)]$, is reflected at the interface; the other part, $\exp(-ik_1 x)$, is transmitted across the interface. In a later chapter we shall discuss these questions more thoroughly.

It should be noted that caution is needeed in the application of the above rule for discontinuity conditions. After all, the discontinuity

conditions are part of the physical problem and cannot be obtained by mathematical considerations alone. However, if the mathematical equations are properly formulated the rule will give the correct discontinuity conditions.

To understand what is meant by a proper formulation, consider the case of one-dimensional electromagnetic wave propagation. In general, two kinds of propagation are possible; they are called transverse electric (TE) and transverse magnetic (TM) propagation. In both TE and TM propagation we must solve the equation

$$u'' + \omega^2 \mu\epsilon\, u = 0,$$

where ω is the frequency of the wave, μ the magnetic permeability, and ϵ the dielectric constant of the medium. Suppose we are interested in the transmission of a wave across the interface of two media for which ϵ has a constant value, but different in each medium, while the constant value μ is the same for both media. For TE propagation the discontinuity conditions are that u and u' be continuous across the interface, while for TM propagation the discontinuity conditions are that u and $\epsilon u'$ be continuous. This later case seems to contradict our rule. However, if we scrutinize the physical situation we observe that for the TE case the equation

$$u'' + \omega^2 \mu\epsilon u = 0$$

is correct even when ϵ varies continuously, but for the TM case the correct equation if ϵ varies continuously is

$$\frac{1}{\epsilon}\,(\epsilon u')' + \omega^2 \epsilon\mu u = 0.$$

If we apply the rule to this latter case we get the correct discontinuity conditions, namely u and $\epsilon u'$ continuous.

From this illustration we deduce the requirement that the mathematical equations should be so formulated that they are valid for continuous changes of the parameters specified in the problem. If they are so formulated, then the rule of this section will give the physically correct discontinuity conditions.

We shall give one final example. Consider the equation

$$u'' + \alpha\, \delta(x-x_0)\, u = 0.$$

What is to be understood by this equation? In order that the term $u(x)$ times the delta function make sense, $u(x)$ must be continuous at $x = x_0$. Integrate the equation from $x_0-\epsilon$ to $x_0+\epsilon$. We get

$$u'\Big|_{x_0-\epsilon}^{x_0+\epsilon} + \alpha u(x_0) = 0;$$

therefore the jump in $u'(x)$ at $x = x_0$ must equal $-\alpha u(x_0)$. For values of $x \neq x_0$, the value of the delta function is zero, and the equation reduces to

$$u'' = 0 .$$

We conclude that the equation is to be interpreted as follows:
The function u is a solution of $u'' = 0$ for all values of x different from x_0. At $x = x_0$, the function u is continuous, but its derivative has a jump of magnitude $-\alpha u(x_0)$. With these specifications the solution of the equation is completely determined as soon as two boundary conditions are given.

Problems:

3.27. Find the Green's function for $Lu = k^2u - u''$ $(-\infty < x < \infty)$ when k is a constant k_1 for $x < 0$, another constant k_2 for $x > 0$ and when the boundary conditions are that u vanish at both $\pm \infty$.

3.28. Find a solution of $u'' + k^2u = 0$ such that $u = e^{-ik_1x}$ for large negative values of x, given that k is a constant k_1 for $x < 0$, another constant k_2 for $0 < x < a$, and again the constant k_1 for $x > a$.

3.29. Find u(x) satisfying $u'' + \alpha\delta(x-x_0)\, u = 0$.

Singular Points of a Differential Equation

During the previous discussion of differential equations we have always made the assumption that $p(x) \neq 0$, because, as was pointed out in an earlier section, if $p(x) = 0$ then some of the solutions of the equation

$$-(pu')' + qu = 0$$

may become infinite at the points where $p(x) = 0$. However, since in many physical applications p(x) does become zero at some point in the interval, it is important to investigate the behavior of the solutions of the differential equation at such points.

In this section we shall discuss the singularities of differential equations and determine the nature of the solutions of the homogeneous equation at such points . In a later section we shall investigate the behavior of the solutions of the inhomogeneous equation. We shall find that just as in the theory of power series, in order to give a complete treatment of the nature of the singularity we must consider the equation and its solutions not only for real values of x, but also for complex values.

Consider then the n-th-order differential equation

$$p_o(x)u^{(n)}(x) + p_1(x)u^{(n-1)}(x) + \ldots + p_{n-1}(x)u'(x) + p_n(x)u = 0,$$

where $p_o, p_1, \ldots, p_n$ are single-valued analytic functions of x in some region R of the complex plane. Rewrite the equation as follows:

$$u^{(n)} + p_1/p_o u^{(n-1)} + \ldots + p_{n-1}/p_o u' + p_n/p_o u = 0 .$$

Any point x at which all the function $p_1/p_o, p_2/p_o, \ldots, p_n/p_o$ are regular, that is, have no singularities, is called an ordinary point of the differential equation. It can be proved† that given n arbitrary initial conditions at an ordinary point x_o, there exists a function u(x) regular in the neighborhood of x_o which satisfies the differential equation and the initial conditions.

A point x at which at least one of the functions $p_o p_1, p_2/p_o, \ldots, p_n/p_o$ has a singularity is called a singular point of the differential equation. For example, consider the equation

$$xu'' + u' = 0$$

which has x = 0 as a singular point. The solutions of this equation are

$$u = \alpha + \beta \log x$$

where α and β are arbitrary constants. We notice that some of the solutions of this differential equation have a branch-point singularity at x = 0, the singular point of the equation. The same situation arises in the case of an arbitrary n-th-order equation, namely some of the solutions of the differential equation may have a singularity at the singular points of the differential equation. All possible types of singularities may occur in the solutions: poles, branch points, and essential singularities such as that of the function $e^{-1/x}$ at x = 0.

We shall distinguish between singular points according to the possible types of singularities the solutions of the differential equation may have there. A singular point is said to be regular if all the solutions of the differential equation have at most a pole or a branch point singularity at that point. A singular point at which some solutions have essential singularities is called irregular.

† For a more detailed discussion see Whittaker and Watson, Modern Analysis (Cambridge Press) Chap. VII and Ince, Ordinary Differential Equations, (Dover, 1944) Chap. XVI.

As an illustration of a regular singular point, consider Bessel's differential equation

$$u'' + \frac{1}{x}u + \left(1 - \frac{\nu^2}{x^2}\right)u = 0 \ . \tag{3.77}$$

We show that the point $x = 0$ is a singular point of this equation. Assume that

$$u = x^{\alpha}(a_0 + a_1 x + a_2 x^2 + \ldots) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \ , \tag{3.78}$$

where $\alpha, a_0, a_1, \ldots$ are constants which will be determined to make (3.78) satisfy (3.77). Note that we may assume $a_0 \neq 0$. For if $a_0 = a_1 = \ldots = a_{k-1} = 0$ but $a_k \neq 0$, then we may write

$$u = x^{\alpha+k}(a_k + a_{k+1}x + \ldots),$$

and replacing $\alpha + k$ by α, a_k by a_0, a_{k+1} by a_1, and so on, we get exactly (3.78).

We have

$$u' = \sum_{n=0}^{\infty} (n+\alpha)\, a_n x^{n+\alpha-1} \ , \quad u'' = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} \ .$$

Substituting these expressions in (3.77) we get

$$\sum (n+\alpha)(n+\alpha-1)\, a_n x^{n+\alpha-2} + \sum_{0}^{\infty} (n+\alpha) a_n x^{n+\alpha-2} + \sum_{0}^{\infty} a_n x^{n+\alpha} - \nu^2 \sum_{0}^{\infty} a_n x^{n+\alpha-2} = 0$$

or

$$\sum_{0}^{\infty} \left[(n+\alpha)^2 - \nu^2\right] a_n x^{n+\alpha-2} + \sum_{0}^{\infty} a_n x^{n+\alpha} = 0.$$

Replace

$$\sum_{0}^{\infty} a_n x^{n+\alpha} \quad \text{by} \quad \sum_{2}^{\infty} a_{n-2} x^{n+\alpha-2} \ ;$$

then we have

$$\sum_{0}^{\infty} \left[(n+\alpha)^2 - \nu^2\right] a_n x^{n+\alpha-2} + \sum_{2}^{\infty} a_{n-2} x^{n+\alpha-2} = 0.$$

Since this equation must hold for every value of x, it follows that the coefficient of every power of x vanishes, that is,

$$\left[(n+\alpha)^2 - \nu^2\right] a_n + a_{n-2} = 0, \quad n = 2, 3, \cdots$$

$$\left[(1+\alpha)^2 - \nu^2\right] a_1 = 0$$

$$(\alpha^2 - \nu^2)\, a_0 = 0.$$

Since $a_0 \neq 0$, the last equation implies that $\alpha = \pm \nu$. Assume $\alpha = \nu$; then the next to last equation implies $a_1 = 0$ and the other equations may be written as follows:

$$a_n = - \frac{a_{n-2}}{n(n+2\nu)}, \qquad n = 2,3 \ldots$$

This recurrence relation determines all the other coefficients in (3.78). We have

$$a_2 = \frac{-a_0}{2\cdot 2(1+\nu)}, \quad a_4 = \frac{-a_2}{4\cdot 2(2+\nu)} = \frac{a_0}{4\cdot 2\cdot 2\cdot 2(2+\nu)(1+\nu)}, \cdots$$

$$a_3 = a_5 = a_7 = \ldots = 0.$$

Consequently, the infinite series

$$u_1 = x^{\nu}\left(1 - \frac{x^2}{2\cdot 2(1+\nu)} + \frac{x^4}{4\cdot 2\cdot 2\cdot 2(2+\nu)(1+\nu)} - \ldots\right)$$

will satisfy (3.77) formally. By the use of the ratio test it can be shown that this series converges for all values of x and hence it is actually a solution of (3.77).

Similarly, if we assume $\alpha = -\nu$, we find that another solution of (3.77) is given by the infinite series

$$u_2 = x^{-\nu}\left(1 - \frac{x^2}{2\cdot 2(1-\nu)} + \frac{x^4}{4\cdot 2\cdot 2\cdot 2(2-\nu)(1-\nu)} - \ldots\right).$$

Since these two solutions are linearly independent, the general solution of (3.77) is

$$u = \alpha u_1 + \beta u_2$$

except when $\nu = 0$ or $\nu = m$, m being an integer. When $\nu = 0$, the infinite series u_2 is exactly the same as u_1, while when $\nu = m$ the expressions for $a_{m+1}, a_{m+2}, \ldots$ in the series for u_2 become infinite.

It is easy to handle the case $\nu = 0$. The equation now is

$$u'' + \frac{1}{x} u' + u = 0. \tag{3.79}$$

Since the last term has no derivatives and is regular at $x = 0$, we expect that the solutions of (3.79) will behave near $x = 0$ like the solutions of the equation

$$u'' + \frac{1}{x} u' = 0.$$

The general solution of this equation is $u = \alpha + \beta \log x$; consequently we expect that one solution of (3.79) will behave like a constant at $x = 0$ and another solution will behave like $\log x$. The series u_1 that we found previously behaves like a constant. Now we must find a solution of (3.79)

which behaves like log x. Assume

$$u_2 = \log x(a_o + a_1 + a_2x^2 + \ldots) + (b_o + b_1x + b_2x^2 + \ldots) ,$$

where $a_o, a_1, a_2, \ldots, b_o, b_1, b_2, \ldots$, are constants to be determined so that u_2 satisfies (3.79). This can be done, but we shall not go into the details. We leave them as an exercise for the reader.

The case where ν equals an integer is not as easy to see through. In the next section we shall discuss the singularities of a system of n first-order differential equations, and from the results obtained there we shall see that when ν equals an integer, the second solution of (3.77) must also contain logarithmic terms. We shall give further details later.

As an illustration of an irregular singular point, consider the equation

$$u'' + k^2u = 0$$

at $x = \infty$. The solutions of this equation are $u = \alpha e^{ikx} + \beta e^{-ikx}$, which have essential singularities at $x = \infty$. The singular point can be transformed into the origin by making the change of independent variable $x = 1/t$. The differential equation now becomes

$$t^2 \frac{d}{dt}\left(t^2 \frac{du}{dt}\right) + k^2u = 0, \tag{3.80}$$

and this has an irregular singular point at $t = 0$. Note that if we assume

$$u = t^{\alpha}(a_o + a_1t + \ldots)$$

we will find that there cannot exist values of $\alpha_o, a_o, a_1, \ldots$ such that the series will satisfy (3.80).

To decide whether a singular point $x = x_o$ of a differential equation is regular or irregular we proceed as follows:

Substitute $(x-x_o)^{\alpha}$ for u in the differential equation and in the resulting equation set the coefficient of the lowest power of $x - x_o$ equal to zero. The equation in α so obtained is called the <u>indicial equation.</u> We state the following Rule. <u>If the degree of the indicial equation is the same as the order of the differential equation, then $x = x_o$ is a regular singular point. If the degree of the indicial equation is less than the order of the differential equation, then $x = x_o$ is an irregular singular point.</u>

We shall illustrate this rule by a few examples. Substitute $u = x^{\alpha}$ into (3.77). The lowest power of x in the equation so obtained is $x^{\alpha-2}$, whose coefficient is $\alpha(\alpha-1) + \alpha - \nu^2 = 0$. Since the degree of the indicial equation is two, which is also the order of the differential equation, we conclude that the point $x = 0$ is a regular singular point for (3.77). For our second example we take equation (3.80). If we put $u = t^{\alpha}$ into (3.80), we find that the indicial equation is $k^2 = 0$. If k is not zero, this equation is impossible, and therefore the point $t = 0$ is an irregular singular point for (3.80). As our final example we consider the equation

$$(x^2u')' + u' = 0 .$$

The indicial equation is now $\alpha = 0$. Since this is an equation of first degree while the differential equation is of the second order, we conclude again that $x = 0$ is an irregular singular point.

Note that to determine whether infinity is a regular or irregular singular point, we replace x by 1/t and use the rule to test whether the point zero is a regular or irregular singular point.

We shall not give a complete proof of this rule, but we shall state a theorem which partly proves the sufficiency of the rule. The proof of this theorem will follow from Theorem III-1 of Appendix III.

Theorem X. If α_o is a root of the indicial equation such that $\alpha_o + m$ is not a root (m is any positive integer), then the differential equation has a solution of the following form:

$$u = x^{\alpha_o}(a_o + a_1x + a_2x^2 + \ldots) .$$

Problems:

3.30. Expand the solutions of the hypergeometric equation

$$x(1-x)\,u'' + \left[\gamma - (\alpha+\beta+1)\,x\right]u' - \alpha\beta u = 0$$

in power series about the origin.

3.31. Expand the solutions of Legendre's equation

$$\left[(1-x^2)u'\right]' + \nu u = 0$$

in power series around the point $x = 1$. Hint. Put $x = 1+t$ in the differential equation. Assume $u = t^{\alpha}(1+a_1t+a_2t^2 + \ldots)$, and continue as for Bessel's equation.

3.32. Consider the equation

$$u'' + q_1 u' + q_2 u = 0$$

where $q_1(x)$, $q_2(x)$ are each the quotients of two power series in x. Show that the indicial equation at x = 0 has degree two if and only if $xq_1(x)$ and $x^2 q_2(x)$ are both bounded as x approaches zero.

3.33. Consider the equation $u'' + q_1 u' + q_2 u = 0$. Suppose that in the neighborhood of infinity, q_1 and q_2 can be expanded in power series in 1/x. What conditions on the power series will imply that infinity is a regular singular point of the differential equation?

Illustrative Example - Logarithmic Singularity

In case the conditions of Theorem X are not satisfied, that is, if $x_0 = m$ is also a root of the indicial equation, then the solutions of the differential equation may have logarithmic singularities. The general theory of such cases is discussed in Appendix III. We shall illustrate this theory by finding expansions near x = 0 for the solutions of the equation

$$u'' + \frac{1}{x} u' + \left(1 - \frac{1}{x^2}\right) u = 0 . \tag{3.81}$$

Note that this is equation (3.77) for the special case $\nu = 1$.

The indicial equation is

$$\alpha(\alpha-1) + \alpha - 1 = 0 \quad \text{or} \quad \alpha^2 = 1.$$

We have already obtained the solution associated with $\alpha = 1$:

$$u_1 = x\left(1 - \frac{x^2}{2\cdot 2\cdot 2} + \frac{x^4}{4\cdot 2\cdot 2\cdot 2\cdot 3\cdot 2} - \dots\right) .$$

We shall now find the solution associated with $\alpha = -1$.

In accordance with the theory discussed in Appendix III, we assume

$$u = x^{-1}(a_0 + a_1 x + a_2 x^2 + \dots) + u_1(x) \log x$$

$$= \sum_0^\infty a_n x^{n-1} + \log x \sum_0^\infty b_n x^{n+1}.$$

We have

$$u'' = \sum_0^\infty (n-1)(n-2)a_n x^{n-3} + \log x \sum_0^\infty (n+1)nb_n x^{n-1} + \sum_0^\infty (2n+1)b_n x^{n-1}$$

$$u' = \sum_0^\infty (n-1)a_n x^{n-2} + \log x \sum (n+1)b_n x^n + \sum b_n x^n.$$

Substituting these expressions in the differential equation, we get

$$\sum_0^\infty (n-1)(n-2)a_n x^{n-3} + \log x \sum_0^\infty (n+1)nb_n x^{n-1} + \sum_0^\infty (2n+1)b_n x^{n-1}$$

$$+ \sum_0^\infty (n-1)a_n x^{n-3} + \log x \sum_0^\infty (n+1)b_n x^{n-1} + \sum_0^\infty b_n x^{n-1}$$

$$+ \sum_0^\infty a_n x^{n-1} + \log x \sum_0^\infty b_n x^{n+1} - \sum_0^\infty a_n x^{n-3} - \log x \sum_0^\infty b_n x^{n-1}$$

$$= \sum_0^\infty x^{n-3}\left[n(n-2)a_n + a_{n-2} + (2n-2)b_{n-2}\right]$$

$$+ \log x \sum_0^\infty x^{n-1}\left[n(n+2)b_n + b_{n-2}\right] = 0.$$

Since this must be identically zero for all values of x, the coefficients of $x^{n-1}\log x$ and of x^{n-3} must be zero. This gives the following set of equations for a_n and b_n:

$$n(n+2)b_n + b_{n-2} = 0,$$

$$n(n-2)a_n + a_{n-2} + (2n-2)b_{n-2} = 0.$$

These equations are to hold for n = 0,1,2,... Note that we have assumed $b_{-1} = b_{-2} = a_{-1} = a_{-2} = 0$.

It is clear that all the odd-numbered coefficients, both a_{2n+1} and b_{2n+1}, are zero. We now determine the even-numbered coefficients. Put n = 2 and we get

$$2 \cdot 4b_2 + b_0 = 0,$$

$$a_0 + 2b_0 = 0.$$

If we assume $b_0 = 1$, then $a_0 = -2$, $b_2 = -\frac{1}{2\cdot 4}$. Put $n = 4$ and the equations become

$$4\cdot 6\, b_4 + b_2 = 0,$$

$$4\cdot 6\, a_4 + a_2 + 6b_2 = 0.$$

The first equation determines b_4, but the second equation determines neither a_4 nor a_2. In fact we may prescribe a_2 arbitrarily, determine a_4 as $-(a_2+6b_2)/4\cdot 2$, and then calculate successively all the other coefficients in the expansion of $u(x)$.

This shows that the expansion of $u(x)$ is not unique since we can always add to $u(x)$ an arbitrary multiple of $u_1(x)$ ans still have a solution of (3.81) whose expansion starts with a term in x^{-1}. In practice, this arbitrary multiple of $u_1(x)$ is so determined that the solution $u(x)$ will have certain properties, for example a given behavior as x approaches infinity. Meanwhile, we assume $a_2 = 0$ and obtain finally the expansion

$$u_2 = x^{-1}(-2 + \frac{6x^4}{2\cdot 4\cdot 2\cdot 4} - \ldots) + x \log x(1 - \frac{x^2}{2\cdot 4} + \ldots).$$

Problems.

3.34. Find the expansion around $x = 0$ of two linearly independent solutions of the system

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & x^{-1} \\ -x & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

3.35. Find the expansion around $x = 0$ of that solution of

$$u'' + \frac{1}{x} u' + (1 - \frac{n^2}{x^2})\, u = 0$$

which begins with x^{-n}.

Solution at An Irregular Singular Point

In most physical applications of differential equations, and especially in problems dealing with wave propagation, the point at infinity is an irregular singular point. As an illustration, consider the equation

$$u'' + k^2 u = 0,$$

where k is an arbitrary real number. The solutions of this equation are of the form $u = \cos kx$, $u = \sin kx$, or $u = e^{\pm ikx}$. Notice that all these solutions have an essential singularity at $x = \infty$.

We shall consider a more general equation.

$$u'' + (k^2 + q(x))\, u = 0. \tag{3.82}$$

and we shall show that if $q(x)$ is small enough at infinity, or more precisely, if

$$\int^{\infty} |q(x)|dx < \infty, \tag{3.83}$$

then there exist solutions of (3.82) which behave like $e^{\pm ikx}$ at $x = \infty$. This last phrase is to be understood in the following sense:

There exists a function $u_1(x)$ which is a solution of (3.82) and which is such that

$$\lim_{x\to\infty} |e^{-ikx}u_1(x) - 1| = 0.$$

Similarly, there exists a function $u_2(x)$ which is a solution of (3.82) and which is such that

$$\lim_{x\to\infty} |e^{ikx}u_2(x) - 1| = 0.$$

To prove these results substitute $u = ve^{ikx}$ in (3.82) and we get

$$v'' + 2ikv' + qv = 0.$$

If we can now show that this equation has a solution $v_1(x)$ which approaches unity as x goes to infinity, we may put $u_1(x) = v_1(x)e^{ikx}$ and will have established the existence of the function $u_1(x)$. Similarly, by putting $u = ve^{-ikx}$ we can prove the existence of $u_2(x)$.

The method of proof is smilar to that used in the proof of Theorem II-I. We write the differential equation in the form

$$v'' + 2ikv' = -qv$$

and consider the right-hand side as a non-homogeneous term; then by using the Green's function for the left-hand side we shall obtain an integral equation for $v(x)$.

The appropriate Green's function can be found from Theorem VIII if we recall that $v(\infty) = 1$ and $v'(\infty) = 0$. We have

$$v(x) = 1 - \int_{\infty}^{x} \frac{1-e^{-2ik(x-\xi)}}{2ik}\, q(\xi)v(\xi)d\xi. \tag{3.84}$$

So far, these results were obtained on the assumption that there exists a function $u_1(x)$ and a corresponding function $v(x)$. We shall shortly construct

a function $v(x)$ which satisfies (3.84). Then it can be shown by differentiation that $v(x)$ satisfies the differential equation. Since it follows from (3.84) that $v(x)$ converges to unity as x approaches infinity, we will have established the existence of $u_1(x)$.

The existence of a solution $v(x)$ of (3.84) will be established by the iteration method used in proving Theorem II-1. We put

$$v_{n+1}(x) = 1 - \int_{\infty}^{x} \frac{1-e^{-2ik(x-\xi)}}{2ik}\, q(\xi)v_n(\xi)d\xi \qquad (n = 1,2,3\ldots)$$

$$v_1(x) = 1,$$

and we shall show that $v_n(x)$ converges to a limit function. We have

$$v_2(x) = 1 - \int_{\infty}^{x} \frac{1-e^{-2ik(x-\xi)}}{2ik}\, q(\xi)d\xi.$$

Note that because of (3.83) the integral converges for all real values of k; therefore $v_2(x)$ is bounded for all values of x. By induction it is easy to prove in the same way that $v_n(x)$ is bounded for all values of x.

Consider the difference $v_{n+1}(x) - v_n(x)$. We have

$$(3.85) \qquad v_{n+1}(x) = -\int_{\infty}^{x} \frac{1-e^{-2ik(x-\xi)}}{2ik}\, q(\xi)\left[v_n(\xi)-v_{n-1}(\xi)\right]d\xi \qquad n = 2,3,\ldots$$

$$v_2(x)-v_1(x) = -\int_{\infty}^{x} \frac{1-e^{-2ik(x-\xi)}}{2ik}\, q(\xi)d\xi \ .$$

Since $\left|\exp\left\{-2ik(x-\xi)\right\}\right| = 1$ for all real values of k, we conclude that

$$(3.86) \qquad |v_2(x) - v_1(x)| \le \int_{\infty}^{x} \frac{1}{k} \int_{\infty}^{x} |q(\xi)|\, d\xi$$

Put

$$Q(x) = \int_{\infty}^{x} |q(\xi)|d\xi \ ,$$

then (3.86) becomes

$$|v_2(x) - v_1(x)| \le \frac{Q(x)}{k} \ .$$

We now prove by induction that

$$(3.87) \qquad |v_{n+1}(x) - v_n(x)| \le \frac{Q(x)^n}{n!k^n}$$

Suppose this inequality holds for $n = 1,2,\ldots,m-1$; then from (3.85) we get

$$|v_{m+1}(x)-v_m(x)| \leq \int_{\infty}^{x} |q(\xi)| \frac{Q(\xi)^{m-1}}{(m-1)!\,k^m}\, d\xi = \frac{1}{(m-1)!\,k^m} \int_{\infty}^{x} Q(\xi)^{m-1}\, dQ(\xi)$$

$$= \frac{Q(x)^m}{m!\,k^m} .$$

which proves (3.87).

The infinite series

$$v_1(x) + \left(v_2(x)-v_1(x)\right) + \left(v_3(x)-v_2(x)\right) + \ldots \tag{3.88}$$

converges for all values of x because each term is less in absolute value than the corresponding term of the absolutely convergent series

$$1 + \frac{Q(x)}{k} + \frac{Q(x)^2}{2!k^2} + \ldots \frac{Q(x)^n}{n!k^n} + \ldots .$$

Since the partial sums of the series (3.88) are $v_n(x)$, this proves that $v_n(x)$ converges to a limit, which we denote by $v(x)$. It is easy now to show that $v(x)$ satisfies the integral equation (3.84) and that $v(\infty) = 1$. As mentioned above, the existence of the function $u_1(x)$ is an immediate consequence of the existence of $v(x)$.

We state this result as

Theorem XI. If k is real and if

$$\int^{\infty} |q(x)|\, dx < \infty,$$

then there exists a solution $u_1(x)$ of the differential equation

$$u'' + (k^2+q)\, u = 0$$

such that

$$\lim_{x\to\infty} \left[u_1 e^{-ikx} - 1\right] = 0.$$

There also exists a solution $u_2(x)$ such that

$$\lim_{x\to\infty} \left[u_2 e^{ikx} - 1\right] = 0.$$

By making suitable transformations we can apply Theorem XI to much wider class of differential equations than its appearance suggests. For example, consider the differential equation

$$(xu')' + \left(k^2 x - \frac{\nu^2}{x}\right) u = 0. \tag{3.89}$$

If we put $u = x^{-1/2}w$ into (3.89) we get

$$w'' + \left(k^2 - \frac{\nu^2 - 1/4}{x^2}\right) w = 0.$$

From Theorem XI it is clear that this equation has solutions which behave at infinity like linear combinations of e^{ikx} and e^{-ikx}; consequently the solutions of (3.89) behave at infinity like $x^{-1/2}$ times linear combinations of e^{ikx} and e^{-ikx}.

Problems.

3.36. Show that when the substitution $u = v\exp\left[-1/2\int b\,dx\right]$ is made in the differential equation $u'' + b(x)\,u' + c(x)u = 0$, the resulting equation for v does not contain any first derivative. Apply this substitution to the equation $u'' + \frac{n}{x}u' + k^2u = 0$, and then by the use of Theorem XI obtain the asymptotic behavior of u for large values of x.

3.37. Consider the equation $-(pu')' + \alpha qu - \beta wu = 0$, where α and β are constants. Put $v = gu$, $g = (wp)^{1/4}$, $y = \int^x (w/p)^{1/2}\,dx$, and then show that

$$-\frac{d^2v}{dy^2} + (Q-\beta)\,v = 0, \text{ where } Q = \frac{1}{g}\frac{d^2g}{dy^2} + \frac{\alpha q}{w}.$$

3.38. If $\int^\infty |Q|\,dy < \infty$, where Q is the function defined in Problem 3.37, then Theorem XI may be applied to the equation in v. Show that in this case we have the asymptotic formula $u = g^{-1}\exp\left[+ i\beta^{1/2}\int^x (w/p)^{1/2}\,dx\right]$.

3.39. Find the behavior of u for large values of x if $-u'' - \beta x^2u + \alpha u = 0$.
Hint. Use Problem 3.38.

3.40 Find the behavior of u for large values of x if $-u'' = (x+\lambda)u$. Hint. Use Problem 3.39 with $q = 0$, $w = x+\lambda$.

Boundary Conditions at a Singular Point†

In the previous sections the behavior of the solutions of the homogeneous differential equation at a singular point was discussed. What can be said about the solutions of the corresponding non-homogeneous equations? Or, an equivalent question, what kind of boundary conditions may we reasonably impose upon the solutions of non-homogeneous equations? That this question is not trivial may be seen from the following simple illustrative problem:

† A similar discussion is given by F. Rellich, Mathematishe Zeitschrift, 49, p. 702 (1944).

Find a function $u(x)$ such that it is a solution of the equation

$$u'' + \frac{2}{x} u' = 1$$

and such that $u(0) = 1$ and $u(1) = 0$.

We shall prove that it is impossible to find such a function $u(x)$. The solutions of the homogeneous differential equation are 1 and x^{-1}. Since a particular solution of the non-homogeneous differential equation is $x^2/4$, the general solution is

$$u = x^2/4 + a + bx^{-1},$$

where a and b are arbitrary constants. To satisfy the boundary condition $u(0) = 1$ we must have $a = 1$ and $b = 0$, but then $u(1) = 5/4$ and not zero; consequently it is impossible to satisfy both boundary conditions simultaneously.

Of course, the trouble is due to the singularity at $x = 0$. However, the following very similar problem has a solution: find $u(x)$ such that

$$u'' - \frac{1}{x} u' = 1$$

and such that $u(0) = 1$ and $u(1) = 0$. The solution is

$$u = \frac{1}{2} x^2 \log x + 1 - x^2.$$

We wish to investigate this phenomenom in order to be able to distinguish between the cases when a boundary condition can be satisfied and when it can not. For simplicity we shall treat only second-order differential operators, but most of the results can be generalized to n-th-order differential operators and to systems of first-order differential equations.

Any second-order differential equation may be written in the form

$$\frac{1}{w} \left[-(pu')' + qu \right] = f(x). \tag{3.90}$$

We assume that $p(x)$, $q(x)$, and $w(x)$ are piecewise analytic functions of x in the interval $(0,1)$, and we also assume that $p(x)$ and $w(x)$ are non-negative in that interval. We suppose that the differential equation has a singularity at $x = 0$ and that $u_1(x)$ and $u_2(x)$ are solutions of the homogeneous differential equation

$$-(pu')' + qu = 0.$$

We suppose also that $u_1(x)$ and $u_2(x)$ are so normalized that the conjunct of u_1 and u_2 is unity.

Any solution of the homogeneous equation will be expressible in the form

$$u = au_1 + bu_2,$$

where a and b are arbitrary constants. We can determine the values of a and b by considering the conjunct of u with the functions u_1 and u_2 in turn. We have

$$J(u,u_1) = aJ(u_1,u_1) + bJ(u_2,u_1) = -b$$

and

$$J(u,u_2) = aJ(u_1,u_2) + b\,J(u_2,u_2) = a.$$

These equations show that we may define the solution u of the differential equation by means of the values of the conjunct of u with particular solutions of that equation. In many cases it is not necessary to know the particular solutions completely; it is enough to know the first terms in their expansions. For example, consider the differential equation

$$(xu')' + k^2 xu = 0. \tag{3.91}$$

The point x = 0 is a regular singular point. By the theory of the preceding sections we know that there is a solution of the equation (3.91) whose expansion in the neighborhood of x = 0 begins with unity, and that there is another solution whose expansion in the neighborhood of x = 0 begins with log x. Consequently, an appropriate boundary condition at x = 0 for a solution of (3.91) would be that

$$\lim_{x\to 0} J(u,a+b\log x) = 0,$$

or, written out in full, that

$$\lim_{x\to 0} x\left[u\frac{b}{x} - u'(a+b\log x)\right] = \lim_{x\to 0}\left[bu - axu' - bxu'\log x\right] = 0.$$

Here again a and b are arbitrary constants. If we put b = 0, then

$$\lim_{x\to 0} xu' = 0$$

would be the appropriate boundary condition for a solution of (3.91) that is regular at x = 0.

It is interesting to note that the typical boundary condition at a regular point, namely

$$\alpha u + \beta u' = 0,$$

can be put in the form of a conjunct condition. If x_0 is a regular point, then $p(x_0) \neq 0$ and there exist functions $u_1(x)$ and $u_2(x)$ which are solutions of (3.90) for f = 0 and which are also such that

$$u_1(x_o) = 1,\ u_1'(x_o) = 0$$
$$u_2(x_o) = 0,\ u_2'(x_o) = 1\ .$$

If we take $u_o(x) = -\beta u_1(x) + \alpha u_2(x)$, then the typical boundary condition given above can be written as the following conjunct condition:

$$J(u,u_o) = 0.$$

Note again that we do not need to know $u_o(x)$ completely, because this condition is equivalent to the following:

$$\lim_{x \to x_o} J[u,-\beta+\alpha(x-x_o)] = 0.$$

Similar results hold in the neighborhood of an irregular singular point. Consider the equation

$$u'' + (k^2+q)\,u = 0$$

and the solutions $u_1(x)$ and $u_2(x)$ defined in Theorem XI. Any solution u of this equation will be a linear combination of u_1 and u_2; consequently it may be written as $u = \alpha u_1 + \beta u_2$. However, from the results of Theorem XI it is clear that we may define u by the two conjunct conditions

$$\lim_{x \to \infty} J[u,e^{ikx}] = -2ik\beta,$$

$$\lim_{x \to \infty} J[u,e^{-ikx}] = 2ik\alpha.$$

We may express the results of our discussion in a simple form if we introduce the concept of the <u>initial</u> of a solution of a differential equation. We define the initial by considering the different cases. At an ordinary point the initial of a solution is the constant and linear term in the power series expansion of the solution about the ordinary point. At a regular singular point, assume that the indicial equation has roots $\alpha = \alpha_1$ and $\alpha = \alpha_2$; then the differential equation will have one solution u_1 whose expansion begins with x^{α_1} and other solutions u_2 whose expansion begins with x^{α_2}. The general solution of the differential equation will be $u = \beta_1 u_1 + \beta_2 u_2$. We call $\beta_1 x^{\alpha_1} + \beta_2 x^{\alpha_2}$ the initial of the solution. Note that if $\alpha_1 = \alpha_2$, the expansion of u_2 will begin with $x^{\alpha_1} \log x$. In that case the initial of u is $x^{\alpha_1}(\beta_1 + \beta_2 \log x)$. Theorem XI shows that an irregular singularity,

there exist solutions u_1 and u_2 which behave like e^{ikx} and e^{-ikx} respectively. The general solution will be of the form $u = \beta_1 u_1 + \beta_2 u_2$. We call $\beta_1 e^{ikx} + \beta_2 e^{-ikx}$ the initial of the solution. Note that according to the discussion of the two previous sections, the initial of the solution may be found quite easily, even without knowledge of the complete solution. Now, we state the Rule. A suitable boundary condition at $x = x_0$ for the solution u of a homogeneous differential equation is obtained by giving the value at $x = x_0$ of the conjunct of u with the initial of any particular solution of that equation.

Boundary Conditions for Non-Homogeneous Differential Equations

In this section we shall discuss the possible boundary conditions that may be imposed on a non-homogeneous differential equation. The essential reason for the distinction between homogeneous and non-homogeneous equations is that in the homogeneous case it is immaterial how singular the solutions are, while in the non-homogeneous case the solutions are defined by integrals which may not converge at a singular point. Because of this, we shall have to distinguish between the cases where the solutions of the differential equation are integrable and the cases where the solutions are not integrable.

Consider again the equation (3.90):

$$Lu = \frac{1}{w}\left[-(pu')' + qu\right] = f(x)$$

in the neighborhood of $x = 0$. We assume first that $x = 0$ is an ordinary point of the equation. This implies that p, q, and w are regular at $x = 0$ and that $p(0) \neq 0$. In this case it is well known that we may assign arbitrarily the values of $u(0)$ and $u'(0)$, or, what is equivalent, assign the value at $x = 0$ of the conjunct of u with the initials of any two solutions of the homogeneous equation.

Suppose now that $x = 0$ is a singular point, and let $u_1(x)$ and $u_2(x)$ be independent solutions of

$$-(pu')' + qu = 0.$$

We assume that $J(u_1, u_2) = 1$. The Green's function for the operator L on the manifold where $u(0) = u'(0) = 0$ is

$$\left[u_1(x)u_2(\xi) - u_2(x)u_1(\xi)\right] H(x - \xi);$$

hence a particular solution of (3.90) is given by the formula

$$(3.92) \qquad u_o = u_1(x)\int_1^x u_2(\xi)w(\xi)f(\xi)d\xi \; - u_2(x)\int_1^x u_1(\xi)w(\xi)f(\xi)d\xi \; .$$

By differentiation we find

$$u_o' = u_1'(x)\int_1^x u_2(\xi)w(\xi)f(\xi)d\xi \; - u_2'(x)\int_1^x u_1(\xi)w(\xi)f(\xi)d\xi \; .$$

From these formulas it is easy to see that

$$(3.93) \qquad J(u_o,u_1) = -J(u_2,u_1)\ \beta_2(x) = \beta_2(x),$$

$$J(u_o,u_2) = J(u_1,u_2)\ \beta_1(x) = \beta_1(x),$$

where

$$\beta_2(x) = \int_1^x u_1(\xi)w(\xi)f(\xi)d\xi \; ,$$

$$\beta_1(x) = \int_1^x u_2(\xi)w(\xi)f(\xi)\,d\xi \; .$$

We shall now show that if these integrals converge as x approaches zero, we may impose any two conjunct conditions on u(x) at x = 0. For example, suppose that the conditions on the solution of (3.90) are

$$J(u, a_1u_1 + b_1u_2)_{x=0} = c_1 \; ,$$

$$J(u, a_2u_1 + b_2u_2)_{x=0} = c_2 .$$

Any solution of (3.90) may be written in the form

$$u = u_o + \gamma_1u_1 + \gamma_2u_2 \; ,$$

where γ_1 and γ_2 are arbitrary constants. With the help of (3.93) we find that

$$J(u_o+\gamma_1u_1+\gamma_2u_2, a_1u_1+b_1u_2)_{x=0} = a_1\beta_2 + b_1\beta_1 + \gamma_1b_2 - \gamma_2a_1 = c_1,$$

$$J(u_o+\gamma_1u_1+\gamma_2u_2, a_2u_1+b_2u_2)_{x=0} = a_2\beta_2+b_2\beta_1 + \gamma_1b_2 - \gamma_2a_2 = c_2 \; ,$$

where we have put $\beta_1 = \beta_1(0)$, $\beta_2 = \beta_2(0)$.

When these linear equations are solved for γ_1 and γ_2, the formula $u = u_o+\gamma_1u_1+\gamma_2u_2$ gives a function u(x) which satisfies the differential equation (3.90) and also the given conjunct conditions. The equations for γ_1 and γ_2 will have a unique solution if and only if $a_1b_2 - a_2b_1 \neq 0$.

This condition merely expresses the requirement that the functions $a_1u_1+b_1u_2$ and $a_2u_1+b_2u_2$ be linearly independent.

Note again that we may write the conjunct conditions in terms of the initial of the solutions instead of the full solutions.

We must now investigate in what cases the integrals in (3.92) will converge. One special case is of particular importance. Consider the scalar product with $w(x)$ as a weight factor, that is,

$$(u,v) = \int_0^1 u(x)v(x)w(x)dx.$$

Note that with this scalar product, Lu is formally self-adjoint and the integrals up to $x = 0$ in (3.92) are (u_2,f) and (u_1,f). We say that a function u belongs to $\mathcal{L}_2$ if (u,u) is finite. If u and v both belong to $\mathcal{L}_2$ then (u,v) is finite because of Schwartz's inequality. Using this result we conclude that if u_1,u_2, and f belong to $\mathcal{L}_2$, then the integrals in (3.92) exist and any two conjunct conditions can be imposed on u.

If u_1 and u_2 belong to $\mathcal{L}_2$, then since any other solution of the homogeneous equation is a linear combination of u_1 and u_2, that solution also belongs to $\mathcal{L}_2$. We shall say that the operator L is of the <u>limit-circle</u>† type to $x = 0$ if all the solutions of $Lu = 0$ belong to $\mathcal{L}_2$ at $x = 0$. We now state the result of our discussion in the following

Theorem XII. <u>Suppose that the operator L is of the limit-circle type at $x = 0$, and suppose that $f(x)$ belongs to $\mathcal{L}_2$ at $x = 0$; then there exists a function $u(x)$ in $\mathcal{L}_2$ which satisfies the equation</u>

$$Lu = f$$

<u>and which satisfies any two independent conjunct conditions at $x = 0$.</u>

As an illustration, consider the operator

$$Lu = -\frac{1}{x}(xu')' - k^2u.$$

If

$$\int_0^1 x(f(x))^2\,dx < \infty,$$

there exists a solution $u(x)$ of the differential equation

$$-Lu = u'' + \frac{1}{x}u' + k^2u = f$$

which satisfies the two conditions:

† The name and the concept were first introduced by H. Weyl, "Über gewöhnliche differential gleichungen mit Singularitäten und die zugehorigen Entwicklungen willkürlicher Funktionen," Math. Annalen, 68, 220-269 (1910).

$$\lim_{x \to 0} xu' = c_1$$

$$\lim_{x \to 0} [u - u'x \log x] = c_2.$$

Suppose, however, that L is not of the limit-circle type; then not every solution of $Lu = 0$ belongs to $\mathcal{L}_2$. We shall then say that L is of the limit-point† type at $x = 0$. For example if $Lu = -u'' - \frac{3}{x}u' = -\frac{1}{x^3}(x^3u')'$, then the solutions of $Lu = 0$ are $u_1 = 1$ and $u_2 = x^{-2}$. Since

$$(u,u) = \int u^2 x^3 dx,$$

we see that u_2 does not belong to $\mathcal{L}_2$, and that L is of the limit-point type.

Another very important example of a limit-point operator, this time at $x = \infty$, is the following:

(3.94) $$Lu = -u'' - (k^2 + q)u.$$

Note that here no solution of $Lu = 0$ belongs to $\mathcal{L}_2$ over the infinite interval. We shall state a theorem which will enable us to discuss the appropriate boundary conditions for an operator such as (3.94):

Theorem XIII. *If k is real and* $\int^\infty |q|\, dx < \infty$ *and* $\int^\infty |f| dx < \infty$, *then the equation*

(3.95) $$u'' + (k^2 + q)u = f$$

has one and only one solution which approaches zero as x approaches infinity.

The proof is simple. Let u_1 and u_2 be the functions defined in Theorem XV, namely, solutions of

$$u'' + (k^2 + q)u = 0$$

which behave like e^{ikx} and e^{-ikx} respectively for large values of x. Note that

$$J(u_1,u_2) = -2ik .$$

If we use the Green's functions for the boundary conditions $u(\infty) = u'(\infty) = 0$, we find that

$$u = \int_x^\infty [u_1(x)u_2(\xi) - u_2(x)u_1(\xi)]f(\xi)d\xi \,/2ik.$$

Since u_1 and u_2 behave like e^{ikx} and e^{-ikx} for large values of x, they are bounded for all values of x; consequently, $u_1(x)u_2(\xi) - u_2(x)u_1(\xi)$ is bounded for all values of x and ξ . We have then

† Also this name and concept were first introduced by H. Weyl, ibid.

$$|u| \leq C \int_x^\infty |f(\xi)| d\xi \quad ,$$

where C is a suitable constant. Since f is integrable, then as x approaches infinity, the integral

$$\int_x^\infty |f(\xi)| d$$

approaches zero and therefore u(x) also approaches zero.

The function u(x) is a unique solution of the problem because any other solution of (3.95) would be of the form $u + \alpha_1 u_1 + \alpha_2 u_2$, where α_1 and α_2 are arbitrary constants, and it is obvious that such a solution does not approach zero as x goes to infinity.

Note that the theorem does not require the function u(x) to be in $\mathcal{L}_2$. In fact this requirement cannot be satisfied, as may be seen from the equation

$$u'' + k^2 u = \frac{e^{ikx}}{x^{3/2}} \quad .$$

It is easy to show that for large values of x, u behaves like the function $ix^{-1/2}e^{ikx}/k$, which is not in $\mathcal{L}_2$.

As an illustration of Theorem XIII, consider the equation

$$u'' + k^2 u = e^{-x}. \tag{3.96}$$

The only solution of this equation which vanishes for large x is $u = e^{-x}(k^2+1)^{-1}$. We see then that it is impossible to find a solution of (3.96) which vanishes at infinity and also satisfies a boundary condition such as u(0) = 0.

Suppose, however, that k is not real, but has a small positive imaginary part. In that case e^{ikx}, which is a solution of the homogeneous equation corresponding to (3.96), vanishes exponentially as x becomes infinite; therefore, for any value of α the function

$$u(x) = \frac{e^{-x}}{k^2+1} + \alpha e^{ikx} \tag{3.97}$$

is a solution of (3.96) which vanishes at infinity. Now, we may impose a boundary condition such as u(0) = 0; then we would find that $\alpha = -(k^2+1)^{-1}$.

We may clarify this result by noting that when k is real no solution of the homogeneous equation corresponding to (3.96) belongs to $\mathcal{L}_2$. But when k has an imaginary part, one and only one solution of the homogeneous equation belongs to $\mathcal{L}_2$: this solution is either e^{ikx} or e^{-ikx}, depending on whether the imaginary part of k is positive or negative. Consequently,

when k is real, the solution of (3.96) which vanishes at infinity is uniquely determined, and it is impossible to impose any other conditions. But when k has an imaginary part, an arbitrary multiple of e^{ikx} or e^{-ikx} can be added so as to satisfy another boundary condition.

In many practical applications where an equation similar to (3.96) occurs, it is physically very reasonable to assume that k has a small imaginary part. This assumption is equivalent to taking into account the effect of a frictional force in mechanics, finite conductivity in electromagnetic problems, or viscosity or damping terms in acoustic problems. Therefore physical problems which impose a boundary condition on the solution of (3.96) may be solved by assuming k to have a small imaginary part. It is obviously immaterial whether k has a positive or negative imaginary part, but for the sake of convenience we shall always asssume k has a positive imaginary part.

It is often desirable to obtain the physically significant solution (3.97) without making the assumption that k has an imaginary part. In such a case we can no longer require that the solution of (3.96) vanish at infinity. Instead we shall say that u must behave like an outgoing wave at infinity. Here we mean by "outgoing wave" that for k real, u is the limit of solutions of (3.96) which vanish at infinity when k has a positive imaginary part.

We summarize our discussion in

Theorem XIV. *If f(x) and g(x) are integrable from 0 to ∞, then, for real k, the equation*

$$u'' + (k^2 + q)u = f \tag{3.98}$$

has a solution which behaves like an outgoing wave at infinity and satisfies any suitable boundary condition at x = 0.

The proof is based on Theorem XIII which states that there exists a unique solution u_0 of (3.98) which vanishes at infinity. Put

$$u = u_0 + \alpha u_1,$$

where α is an arbitrary constant and u_1 is defined in Theorem XI as that solution of

$$u'' + (k^2 + q)u = 0$$

which behaves like e^{ikx} for large x. The value of α can now be determined so that u will satisfy a suitable boundary condition at x = 0. This proves Theorem XIV.

Problems:

3.41 Investigate suitable boundary conditions at $x = 0$ for the solution of

$$u'' + \frac{n}{x} u' + k^2 u = f(x).$$

3.42 Consider the equation in Problem 3.41 and discuss the behavior of u for large values of x. Hint. Remove the first-derivative term as in Problem 3.36.

Appendix I

Green's Function for a System

We must realize that since the Green's function is the inverse operator and since the operators for systems are not only differential operators but also matrices, we must take for the Green's function not a vector solution of the system, but instead a matrix solution. Consider the system of n first-order differential equations:

$$\frac{dU}{dx} = AU \tag{I.1}$$

with the scalar boundary conditions

$$B_1(U) = B_2(U) = \dots B_n(U) = 0. \tag{I.2}$$

The Green's matrix for this system is defined to be the matrix $G(x,t)$ such that

$$\frac{dG}{dx} - AG = +\delta(x - t)I, \tag{I.3}$$

where I is the identity matrix in n dimensions. Also, each column of $G(x,t)$ must satisfy the boundary conditions (I.2).

It is easy to see that (I.3) is the definition suitable for solving the non-homogeneous equation

$$\frac{dU}{dt} - AU = P, \tag{I.4}$$

where P is a given vector. We find that the vector U defined by the formula

$$U = \int_0^1 G(x,t)P(t)dt$$

is a solution of (I.4) which satisfies the boundary conditions (I.2).

For all values of $x \neq t$, the function $G(x,t)$ satisfies the homogeneous equation (I.1). If we integrate (I.3) with respect to x, we get

$$G - \int_0^x AG\,dx = +H(x-t)I,$$

where $H(x)$ is the Heaviside unit function. This equation implies that the diagonal elements of G have jumps of magnitude 1 at $x = t$, while the non-diagonal elements of G are continuous at $x = t$. This result can be verified by putting

$$G = +H(x-t)I + K(x,t),$$

where $K(x,t)$ is a matrix, and substituting this expression in (I.3). We get

$$\frac{dK}{dx} - AK = M, \tag{I.5}$$

where M is a matrix with piecewise continuous elements. It can be shown that (I.5) has a solution in which the elements of K are all continuous functions, and

consequently the only discontinuities of G must be those due to the term $H(x-t)I$. Since this latter matrix has jumps of magnitude 1 along the diagonal when $x = t$, the discontinuity conditions on G are verified.

We shall now give a formula for the Green's matrix of a system. The formula can be easily transformed to give the Green's function of an n-th-order differential equation.

Theorem I-I. Suppose that $U_1(x),\cdots,U_n(x)$ are n linearly independent vector solutions of the system

$$\frac{dU}{dx} = AU, \tag{I.1}$$

and let $S(x)$ be the matrix whose columns are the vectors $U_1,\cdots,U_n$. If there does not exist a solution U of (I.1) which also satisfies the boundary conditions (I.2), then there exists a matrix C, whose elements are functions of t, such that the matrix

$$G = S(x)C(t) + S(x)S(t)^{-1} H(x-t) \tag{I.6}$$

is the Green's matrix of the differential equation (I.1) with the boundary conditions (I.2).

Note that from the corollary to Theorem IV it follows that $\det S \neq 0$, and so the inverse, $S(t)^{-1}$, exists.

First, it is clear from the definition (I.6) that each column of G is a linear combination of the columns of $\mathcal{S}$, that is, of $U_1,\cdots,U_n$, and therefore each column of G satisfies (I.1) for $x \neq t$. Second, the change in G as x goes across the value $x = t$ is $S(t)S(t)^{-1} = I$. This means that the diagonal elements in G have jumps of magnitude 1, while the non-diagonal elements are continuous. These are the correct conditions for the Green's matrix at $x = t$.

Finally, we must show that C can be determined so that the boundary conditions (I.2) are satisfied by each column of G. Let $c_1(t),\cdots, c_n(t)$ be the elements of the first column of C; then the first column of G will be

$$G_1 = c_1U_1 + \cdots + c_nU_n + V_1H(x-t),$$

where V_1 represents the first column of $S(x)S(t)^{-1}$. Since the boundary conditions are linear and homogeneous, it follows that

$$B_k(G_1) = c_1B_k(U_1)+\cdots+c_nB_k(U_n)+B_k(V_1H(x-t)),\ k=1,2,\ldots,n.$$

These n linear equations for the unknowns $c_1,c_2,\cdots,c_n$ will have a solution if the determinant of the coefficients, namely

$$\det B_k(U_j) \quad (k,j = 1,2,\ldots,n),$$

is not zero. However, if the determinant were zero there would exist constants $\alpha_1, \alpha_2,\ldots,\alpha_n$ such that

$$0 = \alpha_1 B_k(U_1)+\ldots+\alpha_n B_k(U_n)-B_k(\alpha_1 U_1+\ldots+\alpha_n U_n) \quad k = 1,2,\ldots n.$$

This would contradict the hypothesis of Theorem I-I because the vector

$$u = \alpha_1 U_1 + \ldots + \alpha_n U_n$$

would be a solution of (I.1) satisfying (I.2). Consequently, the determinant is not zero and the quantities $c_1,c_2,\ldots,c_n$ can be found. Similarly, we can find the elements of every other column of C and so prove the theorem.

It is easy to show, either by the use of Theorem I-I or by a direct proof, that the Green's function for an n-th order differential equation is given by the following:

Theorem I-II. Suppose that $u_1(x),\ldots,u_n(x)$ are n linearly independent solutions of the homogeneous differential equation

$$(I.7) \qquad Lu = p_o \frac{d^n u}{dx^n} +\ldots+ p_n u = 0,$$

If there does not exist a solution of (I.7) which also satisfies the boundary conditions

$$(I.8) \qquad B_1(u) = \cdots = B_n(u) = 0,$$

then there exist n functions $c_1(t),\ldots,c_n(t)$ such that the function

$$g(x,t) = c_1 u_1(x)+\ldots c_n u_n(x) - \frac{G(x,t)}{p_o(t)W(u_1,\cdots,u_n)} H(x-t),$$

where

$$G(x,t) = \begin{vmatrix} u_1(t) & u_2(t) & \cdot & \cdot & \cdot & u_n(t) \\ u_1'(t) & u_2'(t) & \cdot & \cdot & \cdot & u_n'(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-2)}(t) & u^{(n-2)}(t) & & \cdot & \cdot & u_n^{(n-2)}(t) \\ u_1(x) & u_2(x) & \cdot & \cdot & \cdot & u_n(x) \end{vmatrix},$$

is the Green's function for equation (I.7) and boundary conditions (I.8).

Here $W(u_1,\ldots,u_n)$ is the value of the Wronskian of $u_1(x),\ldots,u_m(x)$ at $x = t$. The proof of this theorem is left to the reader as an exercise.

Problems:

I.1. Find the Green's matrix for Equation (I.1) when

$$A = \begin{pmatrix} o & k \\ k & o \end{pmatrix}$$

and the boundary conditions are that the first component of U equals 0 at $x = 0$ and that the second component of U equals 0 at $x = 1$.

I.2. Find the Green's function for the equation $u^{(iv)} + k^4 u = 0$ in the space defined by the boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$.

I.3. Prove Theorem I-II.

I.4. Let S be the matrix defined in Theorem I-I. Show that

$$\frac{dS}{dx} = SD$$

and that

$$\frac{dS^{-1}}{dx} = -S^{-1}A.$$

Hint. Differentiate the identity $S(x)S(x)^{-1} = I$ with respect to x.

Appendix II

Existence of a Solution of Equation (3.29)

We now wish to justify the statement that the solution of equation (3.29) is a continuously differentiable function of x. We shall prove a theorem which will contain this result as a special case.

Theorem II-I. Let p(x), q(x), and f(x) be piecewise continuous functions of x in the closed interval (0.1), and let p(x) be positive in that interval; then there exists a continuous function u(x) such that pu' exists and is continuous for all x; furthermore

$$(pu')' - qu = f(x) \tag{II-I}$$

for all values of x for which both sides are continuous functions of x; and $u(0) = u'(0) = 0$.

The proof will illustrate how the Green's function can be used to transform a differential equation into an integral equation. (II.1) may be written in the form

$$Lu = (pu')' = qu + f(x) = h(x).$$

Suppose that h(x) were known; then u would be found by using the Green's function for L, just as the solution of (3.36) was given by (3.38).

Consider the equation defining the Green's function, namely

(II.2) $$(pg')' = -\delta(x-t),$$

with $g(0) = g'(0) = 0$. It can be solved by direct integration. We have

$$pg' = -H(x-t) + c$$

where c, the constant of integration, must be zero, since $g'(0) = 0$. Put

$$r(x) = \int_0^x ds/p(s);$$

a second integration will then give

$$g(x) = [r(t)-r(x)]H(x-t).$$

Multiply (II.2) by $-h(t)$ and integrate from 0 to 1. We find that the solution of $Lu = h$ is

(II.3) $$u = \int_0^x [r(x)-r(t)]h(t)dt.$$

Since $h(t) = g(t)u(t)+f(t)$ is not a known function of t, equation (3.84) becomes the following integral equation for $u(x)$:

(II.4) $$u(x) = \int_0^x f(t)[r(x)-r(t)]dt + \int_0^x q(t)u(t)[r(x)-r(t)]dt.$$

We have thus shown that if (II.1) has a solution $u(x)$, this solution will satisfy the integral equation (II.4). We now show the converse. Suppose that (II.4) has a solution $u(x)$. Note that from its definition, $r(x)$ is a continuous function; consequently the right-hand side of (II.4), and therefore also $u(x)$, is a continuous function of x. More, however, is true. When we differentiate the right-hand side of (II.4) we find that

$$u'(x) = \int_0^x f(t)dt/p(x) + \int_0^x q(t)u(t)dt/p(x).$$

This equation shows that for all x such that $p(x)$ is continuous, $u'(x)$ exists and

(II.5) $$p(x)u'(x) = \int_0^x f(t)dt + \int_0^x q(t)u(t)dt.$$

Since the right-hand side of this equation is a continuous function of x for all x, then at the points where $p(x)$ is discontinuous we may define $u(x)'$ so that $p(x)u(x)'$ will be continuous everywhere.

Because the integrands in (II.5) are piecewise continuous functions of x we may differentiate the right-hand side of (II.5); we find that

$$(pu')' = f(x) + q(x)u(x)$$

whenever the right-hand side is continuous. This shows that u satisfies (3.39). From (II,5) it is clear that $u'(0) = 0$. This shows that (II.4) is completely equivalent to (II.1) and its boundary conditions.

In order to complete the proof of Theorem XII we need only show that there exists a solution of equation (II.4). At first glance, this does not seem to be a simplification since equation (II.4) seems more complicated than the original equation (II.1). However, equation (II.4) is actually much better suited for our purpose because it contains an integral operator, which is bounded, while (II.1) contains a differential operator, which is always unbounded.

The technique which we shall use to prove the existence of a solution of (II.4) is exemplified in the following case.

Suppose K is a bounded operator in a complete linear space, and b is a known vector; we wish to find a vector y such that

$$y = b + ky. \tag{II.6}$$

If the bound m of K is less than one, we shall prove that such a vector exists. Replace (II.6) by the following sequence of approximating equations:

$$y_1 = b$$

$$y_{n+1} = b + Ky_n, \qquad n = 1,2,\ldots$$

We have

$$y_{n+1} - y_n = K(y_n - y_{n-1})$$

for all $n > 1$, and therefore

$$|y_{n+1} - y_n| \leq m\,|y_n - y_{n-1}| \leq m^2\,|y_{n-1} - y_{n-2}| \leq \ldots\, m^n|b|.$$

Since

$$y_p - y_q = (y_p - y_{p+1}) + (y_{p+1} - y_{p+2}) + \ldots + (y_{q-1} - y_q),\ p < q$$

we have

$$|y_p - y_q| \leq |y_p - y_{p+1}| + |y_{p+1} - y_{p+2}| + \ldots + |y_{q-1} - y_q|$$

$$\leq (m^p + m^{p+1} + \ldots + m^{q-1})|b| = \frac{m^p - m^q}{1-m}\,|b|.$$

If $m < 1$, then m^p and m^q approach zero as p and q approach infinity; consequently, the limit of $|y_p - y_q|$ is zero. Since the space is complete, the sequence y_p must approach a limit which we denote by y.

If we let n approach infinity in the equation

$$y_{n+1} = b+Ky_n,$$

we find that

$$y = b + Ky;$$

consequently, we have established the fact that there exists a solution of the original equation, namely the vector y, which is the limit of the sequence y_n.

Now let us return to the proof of the existence of a solution of equation (II.4). The function u(x) will correspond to the vector y, the first integral on the right will correspond to the vector b, and the integral operator whose kernel is $q(t)[r(x)-r(t)]$ will correspond to the bounded operator K. We proceed just as in the case of (II.6).

Put

$$u_1(x) = \int_0^x f(t)[r(x)-r(t)]dt,$$

and for n = 1,2,...,put

$$u_{n+1}(x) = \int_0^x f(t)[r(x)-r(t)]dt + \int_0^x q(t)[r(x)-r(t)]u_n(t)dt.$$

We have

$$u_{n+1}(x) - u_n(x) = \int_0^x q(t)[r(x)-r(t)][u_n(t)-u_{n-1}(t)]dt$$

and therefore

$$|u_{n+1}(x)-u_n(x)| \leq \int_0^x |q(t)|\ |r(x)-r(t)|\ |u_n(t)-u_{n-1}(t)|dt.$$

In order to estimate the right-hand side we consider the function

$$(II.8) \qquad R(x) = \int_0^x [r(x)-r(t)]\ dt = \int_0^x dt \int_t^x \frac{ds}{p(s)} = \int_0^x \frac{s}{p(s)}\ ds,$$

where we have used the previously given definition of r(x), namely

$$r(x) = \int_0^x \frac{ds}{p(s)}\ .$$

Since p(s) is positive, it follows that both r(x) and R(x) are non-negative and increasing functions of x. Note that

$$\int_0^x R(t)^k[r(x)-r(t)]dt = \int_0^x dtR(t)^k \int_t^x \frac{ds}{p(s)} = \int_0^x \frac{ds}{p(s)} \int_0^s dtR(t)^k.$$

Since R(t) is an increasing function of x,

$$\int_0^s dt R(t)^k \leq R(s)^k \int_0^s dt = sR(s)^k ;$$

consequently, if we use the fact that from (II.8), $R'(x) = x/p(x)$, we have

$$(II.9) \int_0^x R(t)^k \left[r(x)-r(t)\right] dt \leq \int_0^x \frac{s\,ds}{p(s)} R(s)^k = \int_0^x R'(s)ds R(s)^k = \frac{R(x)^{k+1}}{k+1} .$$

We can now get an estimate of $|u_{n+1}(x)-u_n(x)|$ from (II.7). First, we note that since $p(x)$, $q(x)$, and $f(x)$ are piecewise continuous functions of x, they must be bounded in absolute value; also, since $p(x)$ is positive in the interval $(0,1)$ it is greater than some fixed constant, and consequently $r(x)$ is bounded. Let m be a bound for the functions $p(x)$, $q(x)$,$f(x)$, and $r(x)$; then from the definition we have

$$|u_1(x)| \leq \int_0^x |f(t)| \left[r(x)-r(t)\right] dt \leq m\, R(x),$$

and from (II.7) and (II.9) we find that

$$|u_2(x)-u_1(x)| \leq \int_0^x |q(t)| \left[r(x)-r(t)\right] mR(t)dt$$

$$\leq m^2 \int_0^x R(t)\left[r(x)-r(t)\right] dt \leq \frac{m^2 R(x)^2}{2} .$$

Now we use mathematical induction. Suppose that for $n = 1,2,\ldots,k$,

$$|u_n(x)-u_{n-1}(x)| \leq \frac{m^n R(x)^n}{n!} ;$$

we shall prove the same result holds for $n = k+1$. From (II.7) and (II.9) we have

$$|u_{k+1}(x)-u_k(x)| \leq \int_0^x |q(t)| \left[r(x)-r(t)\right] m^k R(t)^k \, dt/k!$$

$$\leq \frac{m^{k+1}}{k!} \int_0^x R(t)^k \left[r(x)-r(t)\right] dt \leq \frac{m^{k+1} R(x)^{k+1}}{(k+1)!}$$

Consider the infinite series

$$u_1(x) + \left[u_2(x)-u_1(x)\right] + \left[u_3(x)\, u_2(x)\right] + \ldots$$

We have just shown that each term of this series is less in absolute value than the corresponding term of the series

$$mR(x) + \frac{m^2R(x)^2}{2!} + \frac{m^3R(x)^3}{3!} + \ldots = -1 + \exp[mR(x)].$$

This latter series converges absolutely and uniformly for all values of x in (0,1); consequently the original series converges absolutely and uniformly to some function u(x). Since the n-th partial sum of the original series is $u_n(x)$, we have

$$u(x) = \lim u_n(x)$$

as n approaches infinity.

All that remains to be shown is that u(x) satisfies (II.4). This is easy. Consider the equation defining $u_{n+1}(x)$ and let n approach infinity. We find that

$$u(x) = \int_0^x f(t)[r(x)-r(t)]dt + \int_0^x q(t)[r(x)-r(t)]q(t)dt,$$

which is (II.4). Consequently we have proved Theorem II-1.

Results similar to those of Theorem II-1 can be proved for n-th-order differential equations or for systems of first-order differential equations, but we shall not discuss them here.

Problem:

II.1 Suppose that A is a matrix with piecewise continuous elements. Prove by the method of this section that given a constant vector U_o there exists a vector U such that

$$\frac{dU}{dx} = AU$$

and such that the value of U for x = 0 is U_o. For all values of x for which all the elements of A are continuous, the elements of u are piecewise differentiable functions of x.

Appendix III

Singular Solutions of a System of Equations

Consider the system of n first-order equations

$$\frac{dU}{dx} = A(x)U,$$

where A is a matrix whose elements are single-valued analytic functions of x. The point $x = x_o$ is a regular point of the system if it is a regular point for all the elements of A. It is a singular point of the system if it is a singularity of at least one of the elements of A. It can be shown that $x = x_o$ is a regular singular point (in the same sense as defined in the section on singular points of a differential equation) if the singularities of the elements of A are only simple poles.

Suppose for simplicity that $x_0 = 0$ and assume that x_0 is a regular singular point for the system. We may expand each one of the elements of A in a power series around zero and then collect termsaccording to powers of x. In this way we get the following result:

$$\frac{dU}{dx} = (A_0x^{-1}+A_1+A_2x + A_3x^2+ \ldots)U$$

or

$$x\frac{dU}{dx} = (A_0+A_1x + A_2x^2 + \ldots)U = (A_0 + xB)U, \tag{III.1}$$

where $A_0, A_1, A_2, \ldots$ are matrices whose elements are constants and B is a matrix whose elements are regular functions at $x = 0$.

Let us first consider the simpler system

$$x\frac{dU}{dx} = A_0U . \tag{III.2}$$

If we make the change of independent variable $x = e^t$, (III.2) becomes

$$\frac{dU}{dt} = A_0U.$$

We know that for each eigenvalue λ of the matrix A_0 this system has a vector solution with components $e^{\lambda t}$. If λ is an eigenvalue of rank ν, then the system also has a solution with components such as

$$e^{\lambda t}\left[t^\nu + a_1t^{\nu-1} + \ldots + a_\nu\right] .$$

Writing these results in terms of x, we see that (III.2) has a solution with components x^λ or $x^\lambda\left[(\log x)^\nu + a_1(\log x)^{\nu-1} + \ldots + a_\nu\right]$.

Let us now return to equation (III.1). We assume that the vector U can be expanded in a power series as follows:

$$U = x^\alpha(U_0 + xU_1 + x^2U_2 + \ldots). \tag{III.3}$$

Here $U_0, U_1, U_2, \ldots$ are vectors whose components are constants. Note that we may assume U_0 to be not identically zero.

Substituting the power series for U into (III.1) we get

$$\alpha x^\alpha U_0 + (\alpha+1)x^{\alpha+1}U_1 + (\alpha+2)x^{\alpha+2}U_2 + \ldots = x^\alpha A_0U_0 + x^{\alpha+1}(A_0U_1 + BU_0) + x^{\alpha+2}(A_0U_2+BU_1) + \ldots$$

When we equate the coefficients of corresponding powers of x, we obtain the following set of vector equations:

$$A_0U_0 = \alpha U_0$$

$$A_0U_1 = (\alpha + 1)U_1 - A_1U_0$$

$$A_0U_2 = (\alpha + 2)U_2 - A_1U_1 - A_2U_0.$$

Each vector equation is really a set of n linear equations for the n components of the corresponding vector. Since the first vector equation is homgeneous, it will have a non-zero solution if and only if the determinant of $A_0 - \alpha$ is zero. This implies that α is an eigenvalue of the matrix A_0, and then U_0 will be an eigenvalue of A_0.

Let U_0 be any eigenvalue of A_0 corresponding to the eigenvector α. Write the second equation as follows:

$$(A_0 - \alpha - 1)U_1 = A_1U_0.$$

It is clear that this equation can be solved for U_1 if the determinant of $A_0 - (\alpha + 1)$ is not zero. Similarly the $(k + 1)$-th equation,

$$(A_0 - \alpha - k)U_k = -A_1U_{k-1} - A_2U_{k-2} - \dots - A_kU_0,$$

can be solved for U_k if the determinant of $A_0 - (\alpha + k)$ is not zero. In this way we may obtain successively the vectors $U_1, U_2, \dots$, and then the series (III.3) is a formal solution of the original system. It can be proved that the infinite series converges in some neighborhood of the origin and therefore the infinite series is an actual solution of the system in that region.

It is clear that the above argument is valid for any eigenvector of A_0. Suppose, however, that A_0 has an eigenvector of rank more than one. What kind of solution, if any, will it correspond to? To make matters concrete, let us assume that α is an eigenvalue of rank two for A_0, and let us assume that

$$(III.4) \qquad u = x^{\alpha}\log x(U_0 + xU_1 + x^2U_2 + \dots) + x^2(V_0 + xV_1 + x^2V_2 + \dots).$$

Substituting this expression in (III.1) and equating first the coefficients of $x^{\alpha+n}\log x$ and then the coeffients of $x^{\alpha+n}(n = -1,0,1,2,\dots)$, we get

$$
\begin{aligned}
A_0U_0 &= \alpha U_0 \\
A_0U_1 &= (\alpha + 1)U_1 - A_1U_0 \\
A_0U_2 &= (\alpha + 2)U_2 - A_1U_1 - A_2U_0 \\
&\cdot \qquad\quad \cdot \qquad\quad \cdot
\end{aligned}
$$

(III.5)

$$
\begin{aligned}
A_0V_0 &= \alpha V_0 + U_0 \\
A_0V_1 &= (\alpha + 1)V_1 + U_1 - A_1V_0 \\
A_0V_2 &= (\alpha + 2)V_2 + U_2 - A_1V_1 - A_2V_0. \\
&\cdot \qquad\quad \cdot \qquad\quad \cdot
\end{aligned}
$$

Consider the first equation of the second set, namely

$$(A_0 - \alpha)V_0 = U_0.$$

Since α is an eigenvalue of A_0, the determinant of $A_0 - \alpha$ vanishes and consequently we cannot conclude that this non-homogeneous equation must have a solution. However, we have assumed that α is an eigenvalue of rank two. Let V_0 be the corresponding eigenvector of rank two; then by definition

$$(A_0 - \alpha)^2V_0 = 0$$

while

$$(A_0 - \alpha)V_0 \neq 0.$$

Denote the vector $(A_0 - \alpha)V_0$ by U_0. We see that the first equation of the first set in (III.5) will be identically satisfied since

$$(A_0 - \alpha)U_0 = (A_0 - \alpha)^2V_0 = 0.$$

Note that the first set of equations obtained by assuming (III.4) is exactly the same as the set we obtained by assuming (III.3). Let $\bar{U}(x)$ denote the solution obtained previously by assuming (III.3); then from (III.4) we have

$$U = \bar{U}(x)\log x + x^{\alpha}(V_0 + xV_1 + x^2V_2 + \ldots),$$

where $V_0, V_1, V_2, \ldots$ satisfies the second set of equations in (III.5).

Since we have assumed that the determinant of $A_0 - (\alpha + k)$ does not vanish, we may solve successively the remaining equations of the second set to obtain $V_1, V_2, \ldots$. In this way we obtain a solution of the form (III.4) corresponding to any eigenvector of rank two. In the same way can show that if α is an eigenvalue of rank m then

$$U = \bar{U}_0(\log x)^{m-1} + \bar{U}_1(\log x)^{m-2} + \ldots + U_{m-1}$$

is a solution of the system of differential equations. Here $\bar{U}_o$, $\bar{U}_1$,..., $\bar{U}_{m-1}$ are vectors whose components are regular functions of the form x^α multiplied by a power series in x. Note also that $\bar{U}_o \log x^{m-2} + \bar{U}_1 (\log x)^{m-3} + \ldots \bar{U}_{m-2}$ will be a solution of the system.

We must now consider what happens when both α and $\alpha+k$ are eigenvalues of A_o (k a fixed positive integer). In this case the previous arguments break down because the determinant of $A_o-(\alpha+k)$ is zero. We expect that the solution will contain logarithms and we shall see that this is actually the case.

Since $\alpha+k$ is an eigenvalue of A_o, there exists a vector

$$x^{\alpha+k}\,\bar{U}^{(k)} = x^{\alpha+k}(W_o + xW_1 + x2W_2 + \ldots)$$

which is a solution of the system.

Here $W_o, W_1, W_2 \ldots$ are vectors whose components are constants. Note that W_o is an eigenvector of A_o corresponding to the eigenvalue $\alpha+k$. We assume that

$$U = x^\alpha(U_o + xU_1 + \ldots + x^{k-1}U_{k-1} + x^kU_k + \ldots) + cx^{\alpha+k}\,\bar{U}^{(k)} \log x, \tag{III.6}$$

where c is an arbitrary constant. If we substitute this expression in (III.1) we obtain the equations

$$\begin{aligned}
A_oU_o &= \alpha U_o \\
A_oU_1 &= (\alpha+1)U_1 - A_1U_o \\
A_oU_k &= (\alpha+k)U_k - A_1U_{k-1} - \ldots - A_kU_o + cW_o.
\end{aligned}$$

The first k of these equations can be solved as before. Write the (k+1)-th equation as follows:

$$(A_o - \alpha - k)U_k = -A_1U_{k-1} - \ldots - A_kU_o + cW_o\,. \tag{III.7}$$

Since the determinant of $A_o-\alpha-k$ vanishes, this equation will have a solution if and only if the right-hand side is orthogonal to all solutions of the adjoint homogeneous equation.

Suppose that W_o is the only eigenvalue of A_o corresponding to the eigenvalue $\alpha+k$; then for that same eigenvalue there exists a unique eigenvector W_o^* of the adjoint matrix A_o^*. Now, if we choose c so that

$$c(W_o^*, W_o) = (W_o^*, A_1U_{k-1} + \ldots + A_kU_o)$$

then the right-handside of (III.7) will be orthogonal to the solutions of the adjoint homogeneous equation and therefore there will exist a vector U_k satisfying (III.7).

The remaining equations for U_{k+1}, U_{+2}, ... can be solved as before, since by assumption $\det[A_o - \alpha - k - m]$ $(m = 1,2,\ldots)$ does not vanish. In this way we have obtained a series of the form (III.6) which is a formal solution of the differential system. Again, it can be proved that this series will converge in some neighborhood of the origin and therefore it will actually be a solution.

We combine our results in the following theorem:

TheoremIII-1. Consider the following system of first-order differential equations:

$$x \frac{dU}{dx} = (A_o + xB)U, \tag{III.8}$$

where A_o is a matrix whose elements are constants and B is a matrix whose elements are regular functions of x in the neighborhood of $x = 0$. Let α be an eigenvalue of A_o such that $\alpha + m$, (m any positive integer) is not an eigenvalue of A_o; then, if U_o is any eigenvector of A_o corresponding to the eigenvalue α, (III.8) has a solution $\bar{U}(x)$:

$$\bar{U}(x) = x^{\alpha}(U_o + xU_1 + \ldots). \tag{III.8}$$

If V_o is an eigenvector of rank p corresponding to the eigenvalue α_o (where α_o is such that $\alpha_o + m$ is not an eigenvalue), then there exists a solution of (III.8) of the form

$$\bar{U}_o(x)(\log x)^{m-1} + \bar{U}_1(x)(\log x)^{m-2} + \ldots + \bar{U}_{m-1}(x)(x),$$

where

$$\bar{U}_o(x)(\log x)^{m-2} + \bar{U}_1(x)(\log x)^{m-3} + \ldots + \bar{U}_{m-2}(x)(x)$$

is also a solution of (III.8), and where

$$\bar{U}_{m-1}(x) = x^{\alpha}(V_o + xV_1 + x^2V_2 + \ldots).$$

Let α_o be an eigenvalue of A_o such that $\alpha_o + k$ (k a positive integer) is also a eigenvalue. If U_o is a eigenvector of A_o corresponding to the eigenvalue α, then there exists a solution of (III.8) of the form

$$x^{\alpha}(U_o + xU_1 + \ldots) + x^{\alpha+k}\,\bar{U}_k \log x,$$

where $x^{\alpha+k}\,\bar{U}_k$ is also a solution of (III.8).

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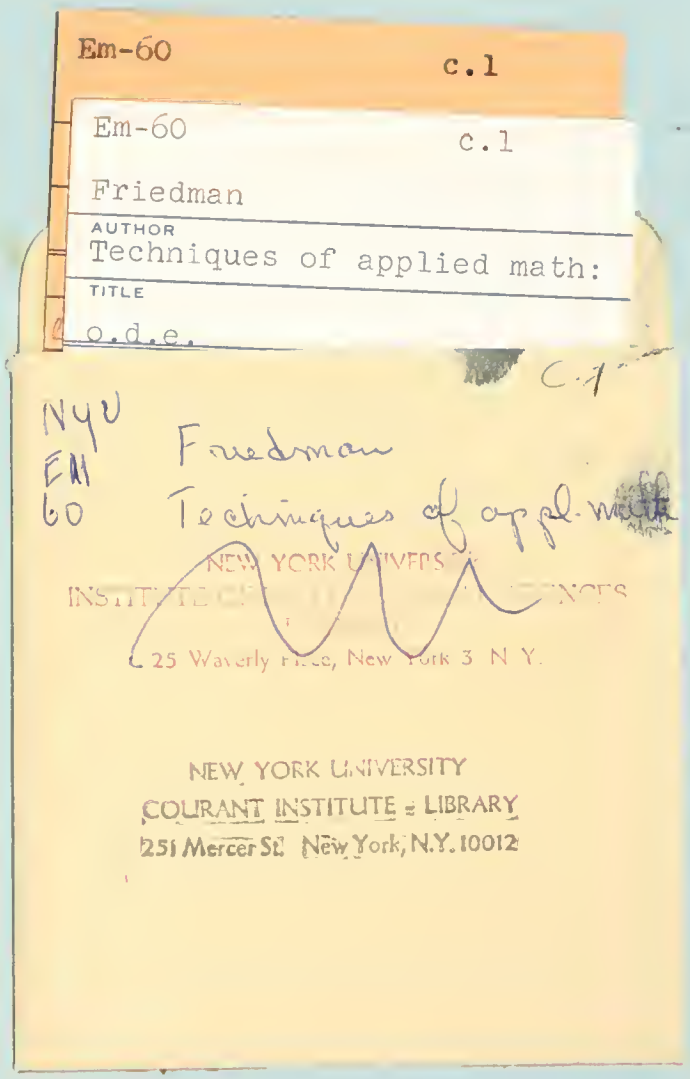

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